MOMENTS AND CHARACTERIZATION OF PROBABILITY DISTRIBUTIONS

ABSTRACT

OF THE

THESIS

SUBMITTED FOR THE AWARD OF THE DEGREE OF

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IN

STATISTICS

BY

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Abstract

Characterization is a condition involving certain properties of a random variables $X=(X_1,X_2,\ldots,X_n)$, which identifies the associated distribution function $F(x)$. The property that uniquely determines $F(x)$ may be based on a function of random variables whose joint distribution is related to that of $X=(X_1,X_2,\ldots,X_n)$. A characterization can be of use in the construction of goodness of fit tests and in the examination of the consequences of modeling assumptions made by an applied scientist. For example, the independence of spacings of order statistics of a random sample from a continuous distribution implies the distribution function is exponential, and thus can be used to construct a goodness-of-fit test, even with a censored sample [Arnold *et al.*, 1998].

The only method of finding distribution function $F(x)$ exactly, which avoids the subjective choice, is a characterization theorem. A theorem is on a characterization of a distribution function if it concludes that a set of conditions is satisfied by $F(x)$ and only by $F(x)$.

Characterization of distributions through linearity of regression of order statistics, record values and generalized order statistics have been considered by many authors.

Ferguson (1967) introduced the characterization of distributions based on the linearity of regression of adjacent order statistics $E(X_{r+1:n} \mid X_{r:n}=x)$ and its dual $E(X_{r:n} \mid X_{r+1:n}=x)$, where $X_{r:n}$ is the r^{th} order statistics. Shanbhag (1970) characterized exponential and geometric distributions in terms of conditional expectations of order statistics for a single order gap.

Khan and Khan (1987) characterized Burr type XII distribution through linear regression of order statistics for a single order gap.
Khan and Abu-Salih (1989) characterized a general class of distributions through conditional expectation of function of order statistics:

$$E[h(X_{r+1:n}) \mid X_{r:n} = x] = a^* h(x) + b^*$$

and

$$E[h(X_{r:n}) \mid X_{r+1:n} = x] = a_1^* h(x) + b_1^*$$

Wesolowski and Ahsanullah (1997) characterized the distributions by the regression of non-adjacent order statistics through the relation

$$E(X_{r+2:n} \mid X_{r:n} = x) = ax + b.$$

Characterization of distributions via linearity of regression of order statistics when gap is higher is considered by Khan and Ali (1987), Franco and Ruiz (1997), Dembińska and Wesolowski (1998) and López-Blázquez and Moreno-Rebollo (1997). Whereas Khan and Abouammoh (2000) extended the result of Khan and Abu-Salih (1989) and characterized the generalized form of distributions through higher order gap. Khan and Athar (2002) also characterized some continuous distributions through linearity of regression when conditioned on a pair of order statistics.

Using the result of Rao and Shanbhag (1994) dealing with an extended version of the integrated Cauchy functional equation, Dembińska and Wesolowski (1998) and Athar *et al.* (2003) characterized the distributions by means of the regression equation

$$E(X_{r+i:n} \mid X_{r:n} = x) = ax + b$$

Characterization of continuous distributions by conditional variance of adjacent order statistics is first considered by Beg and Kirmani (1978). Beg and Kirmani (1978) have shown that

$$V[X_{r+1:n} \mid X_{r:n} = x] = \frac{1}{a^2(n-r)^2}$$

if and only if X has exponential distribution.

Khan and Beg (1987) extended the result and proved that the conditional variance of $X^p_{r+1:n}$ given $X_{r:n} = x$ does not depend on x if and only if X has Weibull distribution.

For record values, Nagaraja (1977) characterized continuous distributions by using the relation

$$E(X_{u(r+1)} \mid X_{u(r)} = x) = ax + b,$$

where $X_{u(r)}$ is the r^{th} record value.

Nagaraja (1988) also characterized distributions by means of

$$E(X_{u(r)} \mid X_{u(r+1)} = x) = ax + b$$

Franco and Ruiz (1996) obtained the distribution function $F(\,)$ from the conditional expectation

$$E[h(X_{u(n-1)}) \mid X_{u(n)} = x]$$

where $h(\,)$ is a real, continuous and strictly monotonic function.

Wesolowski and Ahsanullah (1997) extended the result of Nagaraja (1977) and characterized the distributions for double order gap.

López-Blázquez and Moreno-Rebollo (1997), Dembińska and Wesolowski (2000) and Athar *et al.* (2003) characterized distributions by means of the relation

$$E(X_{u(r+i)} \mid X_{u(r)} = x) = ax + b$$

Gupta and Ahsanullah (2004) characterized distributions through conditional expectation of record values through

$$E[\xi(X_{u(r+2)}) \mid X_{u(r)} = x] = g(x)$$

where $g(x)$ may be non-linear but differentiable *w.r.t.* x. Further Bairamov *et al.* (2005) characterized exponential type distributions via regression on pairs of record values, where regression may not be linear.

Other characterizations results based on conditional expectations of non-adjacent record values are given in Wu and Lee (2001), Raqab (2002) and Wu (2004).

Concept of generalized order statistics (gos) was given by Kamps (1995a). Since many ordered variables like order statistics, record values and $k-$record values are special cases of generalized order statistics, therefore characterization through generalized order statistics is of special interest. Keseling (1999) gave characterization of exponential distribution under the condition

$$E[\psi(X(r+1,n,m,k) - X(r,n,m,k)) \mid X(r,n,m,k) = x] = c$$

where c is a constant and $X(r,n,m,k)$ is the r^{th} *gos*.

Samuel (2008) characterized the distribution function through the relation

$$E[h(X(r+1,n,m,k)) \mid X(r,n,m,k) = x] = a^* h(x) + b^*$$

Ahsanullah and Raqab (2004) proved that the relation

$$E[\psi(X(r+2,n,m,k)) \mid X(r,n,m,k) = x] = g(x)$$

uniquely determines the distribution functions.

Further, Raqab and Abu-Lawi (2004) characterized some general continuous distributions based on conditional expectation

$$E[g(X(r+1,n,m,k)) \mid X(r,n,m,k)=x]=h(x)+c$$

where $h(.)$ and $g(.)$ are real, continuous and strictly increasing functions.

Bienik (2007) characterized continuous distributions based on conditional expectation

$$E[g(X(r,n,m,k)) \mid X(r+1,n,m,k)=x]=h(x),$$

using Meijer's G-function.

Khan and Alzaid (2004) characterized a general class of distribution $\overline{F}(x)=[ax+b]^c$ through linear regression of generalized order statistics using Rao and Shanbhag's (1994) result. They characterized the distributions by means of relation

$$E[X(s,n,m,k) \mid X(r,n,m,k)=x]=a^*x+b^*$$

Khan *et al.* (2006), Beg and Ahsanullah (2006) have characterized the distribution functions through the relation

$$E[\xi\{X(s,n,m,k)\} \mid X(r,n,m,k)=x]=g_{s|r}(x)$$

and its dual

$$E[\xi\{X(r,n,m,k)\} \mid X(s,n,m,k)=x]=g_{r|s}(x)$$

Further Ahsanullah and Beg (2008) characterized the continuous distribution functions conditioned on a pair of adjacent *gos* through the relation

$$E[\xi\{X(r+1,n,m,k)\} \mid X(r,n,m,k)=x,\ X(r+2,n,m,k)=x]=g_{r+1|r,r+2}(x)$$

The present thesis entitled **"Moments and characterization of Probability Distributions"** comprises five chapters, in which Chapter I is introductory in nature and deals with the basic concepts and results needed in the subsequent chapters.

Chapter II deals with the characterization of distributions. These characterization results are based on conditional expectation of record values conditioned on a pair of non-adjacent records.

More specifically, it has been shown here that if

$$g_{j|\,l,\,s}(x,y) = E[h(X_{u(j)})\,|\,X_{u(l)} = x, X_{u(s)} = y]\ ,\ \ l = r, r+1$$

exists, then

$$\frac{\log \overline{F}(x)}{\log \overline{F}(y)} = 1 - e^{-I_1},$$

where,

$$A_1(x,y) = \frac{\frac{\partial}{\partial x} g_{j|\,r,s}(x,y)}{(s-r-1)\,[g_{j|\,r,s}(x,y) - g_{j|\,r+1,s}(x,y)]},\ I_1 = \int_{\alpha}^{x} A_1(t,y)\,dt,$$

and $h(t)$ is a monotonic and differentiable function of t and $g_{j|\,r,\,s}(x,y)$ may not be linear, $\alpha \le x < y \le \beta$.

It has also been shown that if

$$g_{j|\,r,l}(x,y) = E[h(X_{u(j)})\,|\,X_{u(r)} = x, X_{u(l)} = y]\ ,\ \ l = s, s-1,$$

exist, then

$$\frac{1+\log \overline{F}(y)}{1+\log \overline{F}(x)} = 1 - e^{-I_2}\ ,\ \ x < q,$$

where $q \in (\alpha, \beta)$ such that $-\log \overline{F}(q) = 1$,

$$A_2(x,y) = \frac{\frac{\partial}{\partial y} g_{j|\, r,s}(x,y)}{(s-r-1)\,[g_{j|\, r,s}(x,y) - g_{j|\, r,s-1}(x,y)]},$$

and $\quad I_2 = \int_q^y A_2(x,t)\,dt.$

Further if $g(x,y)$ is assumed to be differentiable *w.r.t.* both x and y, the above results can be combined to get

$$\overline{F}(x) = \exp\left[-\frac{e^{I_1}-1}{e^{I_1}+e^{I_2}-1}\right],$$

and,

$$\overline{F}(y) = \exp\left[-\frac{e^{I_1}}{e^{I_1}+e^{I_2}-1}\right]$$

Further a characterizing result has been obtained for the *df*

$$F(x) = 1 - e^{-[ah(x)+b]}, \quad \alpha \le x \le \beta,$$

using these Theorems and some of deductions are discussed.

Chapter III contains result on characterization of continuous distributions conditioned on a pair of non-adjacent generalized orders statistics and then the results are deduced for order statistics and records.

In Chapter IV, a general class of continuous distribution is characterized through the conditional variance of order statistics. Precisely, it has been shown that:

(1) For $0 < r < n$,

$$V[h\{X_{r+1:n}\} \,|\, X_{r:n} = x] = \frac{1}{a^2(n-r)^2}$$

if and only if

$$F(x) = 1 - e^{-ah(x)}$$

where $a > 0$, $h(x)$ is a non-decreasing and twice differentiable function of x such that $h(x) \to 0$ as $x \to \alpha$ and $h(x)\bar{F}(x) \to 0$ as $x \to \beta$.

(2) For $0 < r < n$,

$$V[h\{X_{r:n}\} | X_{r+1:n} = y] = \frac{1}{a^2 r^2}$$

if and only if

$$F(y) = e^{-ah(y)}$$

where $a > 0$, $h(y)$ is a non-increasing and twice differentiable function of y such that $h(y) \to 0$ as $y \to \beta$ and $h(y)F(y) \to 0$ as $y \to \alpha$.

In Chapter V, an explicit expression for the ratio and inverse moments of generalized order statistics from the Burr distribution using hyper-geometric function have been obtained.

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Dedicated to my Parents

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Chairman

Dated: 20.01.2010

CERTIFICATE

This is to certify that the contents in the thesis entitled ***"Moments and Characterization of Probability Distributions"*** by **Mr. Mohd. Jahangir Sabbir Khan** for the award of the degree of Doctor of Philosophy (Statistics) of the Aligarh Muslim University, Aligarh, is a record of bonafide research work, carried out by him under my supervision and guidance.

The thesis has, in my opinion, reached the standard fulfilling the requirements of the Ph.D. degree. The results contained in this thesis have not been submitted to any other University or Institution for the award of any other degree or diploma.

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Jahangir S. Khan

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(Mohd. Jahangir Sabbir Khan)

CONTENTS

Preface

In statistics, generally observations are obtained in terms of random variables $X=(X_1,X_2,...,X_n)$. Now, the task for a statistician is to specify the distribution function $F(x)$ of X. The usual approach is to start with a family $\Im$ of distributions and to select from this family a distribution $F(x)$ which is the most acceptable and plausible one for the given data. Unfortunately, in many cases $\Im$ simply consists of a single function which is dependent on one or several parameters and the observations are used merely to approximate its parameters. The function thus obtained is chosen as $F(x)$.

The only method of finding distribution function $F(x)$ exactly, which avoids the subjective choice, is a characterization theorem. A theorem is a characterization of a distribution function theorem if it concludes that a set of conditions is satisfied by $F(x)$ and only by $F(x)$ [Galambos and Kotz, 1978].

Another important consequence of characterization theorem is that these results help us in better understanding the structures and implications of the choice of distribution for a special problem. With this in view, in this thesis, some distributions are characterized through ordered random variables.

The thesis entitled **"Moments and Characterization of Probability Distributions"** comprises five chapters, in which Chapter I is introductory in nature and deals with the basic concepts and results needed in the subsequent chapters.

Chapter II deals with the characterization of distributions. These characterization results are based on conditional expectation of record values conditioned on a pair of non-adjacent records.

More specifically, it has been shown here that if

$$g_{j|\,l,\,s}(x,y) = E[h(X_{u(j)}) \mid X_{u(l)} = x, X_{u(s)} = y]\ ,\quad l = r, r+1$$

exist, where $X_{u(r)}$ is the r^{th} record value and $g(\,)$ is a finite and differentiable function of x, then

$$\frac{\log \overline{F}(x)}{\log \overline{F}(y)} = 1 - e^{-\int_{\alpha}^{x} A_1(t,y)\,dt}\ ,$$

where,

$$A_1(x,y) = \frac{\frac{\partial}{\partial x} g_{j|\,r,s}(x,y)}{(s-r-1)\,[g_{j|\,r,s}(x,y) - g_{j|\,r+1,s}(x,y)]}\ ,$$

and $h(t)$ is a monotonic and differentiable function of t and $g_{j|\,r,\,s}(x,y)$ may not be linear, $\alpha \le x < y \le \beta$.

Also, it has also been shown that if

$$g_{j|\,r,l}(x,y) = E[h(X_{u(j)}) \mid X_{u(r)} = x, X_{u(l)} = y]\ ,\quad l = s, s-1,$$

exist, where $g(\,)$ is a finite and differentiable function of y, then

$$\frac{1+\log \overline{F}(y)}{1+\log \overline{F}(x)} = 1 - e^{-\int_{q}^{y} A_2(x,t)\,dt}\ ,\quad x < q\,,$$

where $q \in (\alpha, \beta)$ such that $-\log \overline{F}(q) = 1$,

$$A_2(x,y) = \frac{\frac{\partial}{\partial y} g_{j|\,r,s}(x,y)}{(s-r-1)\,[g_{j|\,r,s}(x,y) - g_{j|\,r,s-1}(x,y)]}\,.$$

Further, if $g(x,y)$ is assumed to be differentiable *w.r.t.* both x and y, the above results can be combined to get

$$\overline{F}(x) = \exp\left[-\frac{e^{I_1} - 1}{e^{I_1} + e^{I_2} - 1}\right],$$

and,

$$\overline{F}(y) = \exp\left[-\frac{e^{I_1}}{e^{I_1} + e^{I_2} - 1}\right],$$

where $I_1 = \int_{\alpha}^{x} A_1(t,y)\,dt$, and $I_2 = \int_{q}^{y} A_2(x,t)\,dt$.

As an example, a characterizing result has been obtained for the *df*

$$F(x) = 1 - e^{-[ah(x)+b]}, \quad \alpha \le x \le \beta,$$

using these Theorems and some of deductions are discussed.

Chapter III contains results on characterization of continuous distributions conditioned on a pair of non-adjacent generalized orders statistics and then the results are deduced for order statistics and records.

In Chapter IV, a general class of continuous distribution is characterized through the conditional variance of order statistics. Precisely, it has been shown that:

(1) $V[h\{X_{r+1:n}\} \mid X_{r:n} = x] = \dfrac{1}{a^2 (n-r)^2}$

if and only if

$$F(x) = 1 - e^{-ah(x)}, \quad \alpha < x < \beta,$$

where $X_{r:n}$ is the r^{rh} order statistics, $a > 0$ and $h(x)$ is a non-decreasing and twice differentiable function of x such that $h(x) \to 0$ as $x \to \alpha$ and $h(x)\overline{F}(x) \to 0$ as $x \to \beta$.

(2) $V[h\{X_{r:n}\} \mid X_{r+1:n} = y] = \dfrac{1}{a^2 r^2}$

if and only if

$$F(y) = e^{-ah(y)}, \quad \alpha < y < \beta,$$

where $a > 0$, $h(y)$ is a non-increasing and twice differentiable function of y such that $h(y) \to 0$ as $y \to \beta$ and $h(y)F(y) \to 0$ as $y \to \alpha$.

In Chapter V, an explicit expression for the ratio and inverse moments of generalized order statistics from Burr distribution using hyper-geometric function have been obtained.

In the end, a comprehensive bibliography is given.

Chapter I

PRELIMINARIES AND BASIC CONCEPTS

In this chapter we have introduced those concepts and results, which are used in the subsequent chapters.

1. Order statistics

Let $X_1, X_2, \cdots, X_n$ be independent and identically distributed *(iid)* random variables of size n from a continuous population having probability density function (*pdf*) $f(x)$ and distribution function (*df*) $F(x)$. If these random variables (*rv*) are arranged in ascending order of magnitude such that

$$X_{1:n} \leq X_{2:n} \leq \ldots \leq X_{n:n}$$

then $X_{i:n}$ is called the i^{th} order statistic and $X_{1:n} = \min(X_1, X_2, \cdots, X_n)$ and $X_{n:n} = \max(X_1, X_2, \cdots, X_n)$ are called extremes or smallest and largest order statistics respectively.

Order statistics can also be defined by the means of measurable map $T : \Re^n \to \Re^n$ such that

$$(X_{1:n}, X_{2:n}, \ldots, X_{n:n}) = T(X_1, X_2, \ldots, X_n)$$

(Kamps, 1995 a).

Alternatively, order statistics can also be introduced by the quantile function $\hat{F}_n^{-1}$ of the empirical distribution function

$$\hat{F}_n^{-1}(x) = \frac{1}{n} \sum_{j=1}^{n} I_{(-\infty, x)}(X_j), \; x \in \Re^n \; ,$$

and therefore, order statistics can be introduced as quantiles

$$X_{j:n} = \hat{F}_n^{-1}(\frac{j}{n}),\ 1 \le j \le n\ ,\ \text{(Kamps, 1995)}.$$

The subject of order statistics deals with the properties and applications of these ordered random variable (*rv*) and of function involving them (David and Nagaraja, 2003). It is different from the rank statistics in which the order of the value of the observation rather than its magnitude is considered.

It plays an important role both in model building and in statistical inference. For example, extreme (largest, smallest) values are important in Oceanography (waves and tides), material strength (strength of a chain depends upon the weakest link) and meteorology (extremes of temperature, pressure etc.)

Another very interesting application of order statistics is found in reliability theory. The r^{th} order statistics $X_{r:n}$ in a sample of size n represents the life length of a $(n-r+1)$ out of n system. This system consists of n components of the same kind with independently distributed life lengths. All n components start working simultaneously and the system fails, if r or more component fails. In other words, $n-r+1$ components are necessary for the system to work. For $r=1$ we have a series system and the case $r=n$ corresponds to a parallel system.

1.1. Distribution of the order statistics

The *pdf* of $X_{r:n}$, the r^{th} order statistics is given by (David and Nagaraja, 2003)

$$f_{r:n}(x) = \frac{n!}{(r-1)!\,(n-r)!}[F(x)]^{r-1}[1-F(x)]^{n-r} f(x)\ ,\ -\infty < x < \infty \quad (1.1)$$

The *pdf* of smallest and largest order statistics are,

$$f_{1:n}(x) = n[1 - F(x)]^{n-1} f(x) \ ; \ -\infty < x < \infty \tag{1.2}$$

$$f_{n:n}(x) = n[F(x)]^{n-1} f(x) \ ; \ -\infty < x < \infty \tag{1.3}$$

The *df* of $X_{r:n}$ is given by

$$F_{r:n}(x) = P(X_{r:n} \le x)$$

$$= P(\text{at least } r \text{ of } X_1, X_2, ..., X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} P(\text{exactly } i \text{ of } X_1, X_2, ..., X_n \text{ are less than or equal to } x)$$

$$= \sum_{i=r}^{n} \binom{n}{i} [F(x)]^i [1 - F(x)]^{n-i} \tag{1.4}$$

$$= \frac{n!}{(r-1)!(n-r)!} \int_0^{F(x)} u^{r-1} (1-u)^{n-r} du \tag{1.5}$$

$$= \mathrm{I}_{F(x)}(r, n-r+1) \tag{1.6}$$

where $I_p(a,b) = \dfrac{1}{B(a,b)} \displaystyle\int_0^p t^{a-1} (1-t)^{b-1} dt$,

and

$$B(a,b) = \int_0^1 t^{a-1} (1-t)^{b-1} dt$$

RHS of (1.6) is obtained by the relationship between binomial sums and incomplete beta function. It may be expressed in negative binomial sums as (Khan, 1991).

$$F_{r:n}(x) = \sum_{i=0}^{n-r} \binom{n-1-i}{r-1} [F(x)]^r [1 - F(x)]^{n-r-i}; \qquad -\infty < x < \infty \tag{1.7}$$

For continuous case the *pdf* of $X_{r:n}$ may also be obtained by differentiating (1.5) *w.r.t.* x.

The k^{th} moment of $X_{r:n}$ is

$$\alpha_{r:n}^{k} = E[X_{r:n}^{k}] = \int_{-\infty}^{\infty} x^{k} f_{r:n}(x)\, dx \tag{1.8}$$

The joint *pdf* of $X_{r:n}$, $X_{s:n}$, $1 \le r < s \le n$ is given by

$$f_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!}[F(x)]^{r-1}$$

$$\times [F(y) - F(x)]^{s-r-1}[1 - F(y)]^{n-s} f(x) f(y)\,; -\infty < x < y < \infty \tag{1.9}$$

The joint *df* of $X_{r:n}$ and $X_{s:n}$, $1 \le r < s \le n$ can be obtained as follows:

$$F_{r,s:n}(x,y) = P(X_{r:n} \le x, X_{s:n} \le y)$$

$= P($at least r of $X_1, X_2, \ldots, X_n$ are at most x

and at least s of $X_1, X_2, \ldots, X_n$ are at most $y)$

$= \sum_{j=s}^{n} \sum_{i=r}^{j} P($exactly i of $X_1, X_2, \ldots, X_n$ are at most x

and exactly j of $X_1, X_2, \ldots, X_n$ are at most $y)$

$$= \sum_{j=s}^{n} \sum_{i=r}^{j} \frac{n!}{i!(j-i)!(n-j)!}[F(x)]^{i}[F(y) - F(x)]^{j-i}[1 - F(y)]^{n-j} \tag{1.10}$$

We can write the joint *df* of $X_{r:n}$ and $X_{s:n}$ in (1.10) equivalently as:

$$F_{r,s:n}(x,y) = \frac{n!}{(r-1)!(s-r-1)!(n-s)!} \int_{0}^{F(x)} \int_{0}^{F(y)} u^{r-1}(v-u)^{s-r-1}$$

$$\times (1-v)^{n-s}\, du\, dv$$

$$= I_{F(x),F(y)}(r, s-r, n-s+1)\,; -\infty < x < y < \infty, \tag{1.11}$$

which is incomplete bivariate beta function.

It may be noted that for $x \geq y$

$$F_{r,s:n}(x,y) = F_{s:n}(y) \tag{1.12}$$

The product moment of the j^{th} and k^{th} order of $X_{r:n}$ and $X_{s:n}$ respectively, $(1 \leq r < s \leq n)$ is given by:

$$\alpha_{r,s:n}^{j,k} = E[X_{r:n}^{j} X_{s:n}^{k}] = \iint\limits_{-\infty<x<y<\infty} x^{j} y^{k} f_{r,s:n}(x,y)\,dx\,dy \tag{1.13}$$

In general, the joint *pdf* of $X_{i_1:n}, X_{i_2:n}, \ldots, X_{i_k:n}$ for $1 \leq i_1 < i_2 < \ldots < i_k \leq n$ is given by

$$f_{i_1,i_2,\ldots,i_k:n}(x_{i_1:n}, x_{i_2:n}, \ldots, x_{i_k:n})$$

$$= n! \left\{ \prod_{j=1}^{k} f(x_{i_j}) \right\} \prod_{j=0}^{k} \left\{ \frac{[F(x_{i_{j+1}}) - F(x_{i_j})]^{i_{j+1}-i_j-1}}{(i_{j+1}-i_j-1)!} \right\}$$

$$-\infty < x_{i_1} < x_{i_2} < \ldots < x_{i_k} < \infty \tag{1.14}$$

where $x_0 = -\infty, x_{k+1} = +\infty, i_0 = 0, i_{k+1} = n+1$.

Remarks:

1. The ranking of random variables $X_1, X_2, \ldots, X_n$ is preserved under any monotonic increasing transformation of the random variables.
2. Regarding the probability integral transformation, if $X_{r:n}, 1 \leq r \leq n$, are the order statistics from a continuous distribution $F(x)$, then the transformation $U_{r:n} = F(X_{r:n})$ produces a random variable which is the r^{th} order statistics from a uniform distribution on $U(0,1)$.

3. Even if $X_1, X_2, ..., X_n$ are independent random variables, order statistics are not independent random variables.

4. If $X_1, X_2, ..., X_n$ be *iid* random variables from a continuous distribution, then the set of order statistics $\{X_{1:n}, X_{2:n}, ..., X_{n:n}\}$ is both sufficient and complete (Lehmann, 1986).

5. Let X be a continuous random variable with $E[X_{r:n}] = \alpha_{r:n}$

   a) If $\alpha = E(X)$ exists then $\alpha_{r:n}$ exists, but converse is not necessarily true. That is, $\alpha_{r:n}$ may exist for certain (but not all) values of r, even though α does not exist.

   b) $\alpha_{r:n}$ for all n determine the distribution completely.

2. Truncated distribution

Let X be a continuous random variable having *pdf* $f(x)$ and *df* $F(x)$ in the interval $[-\infty, \infty]$.

$$\text{Let} \quad \int_{-\infty}^{Q_1} f(x)dx = Q \quad \text{and} \quad \int_{-\infty}^{P_1} f(x)dx = P \tag{2.1}$$

where Q_1 and P_1 are known constants. Then doubly truncated *pdf* of X is given by:

$$\frac{f(x)}{P-Q}; \; x \in (Q_1, P_1) \tag{2.2}$$

and the corresponding *df* is given by

$$\frac{F(x)-Q}{P-Q}; \; x \in (Q_1, P_1) \tag{2.3}$$

The lower and upper truncation points are Q_1 and P_1 respectively; the degrees of truncation are Q (from below) and $1-P$ (from above). If we put $Q=0$, the distribution will be truncated to the right. Similarly at $P=1$, the distribution will be truncated to the left, whereas at $Q=0$ and $P=1$, we get the non truncated distribution. Truncated distributions are useful in finding the conditional distributions of order statistics.

In the following, we will relate the conditional distribution of order statistics (conditioned on another order statistic) to the distribution of order statistics from a population whose distribution is truncated from the original population distribution $F(x)$.

Result 2.1 (David and Nagaraja, 2003): Let $X_1, X_2, ..., X_n$ be a random sample from an absolutely continuous population with *df* $F(x)$ and *pdf* $f(x)$ and let $X_{1:n} \le X_{2:n} \le ..... \le X_{r:n} \le ..... \le X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{r:n}$ given that $X_{s:n} = y$ for $s > r$, is the same as the distribution of the r^{th} order statistic obtained from a sample of size $(s-1)$ from a population whose distribution is truncated on the right at y.

Result 2.2 (David and Nagaraja, 2003): Let $X_1, X_2, ..., X_n$ be a random sample from an absolutely continuous population with *df* $F(x)$ and *pdf* $f(x)$, and let $X_{1:n} \le X_{2:n} \le ... \le X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$ given that $X_{r:n} = x$ for $r < s$, is the same as the distribution of the $(s-r)^{th}$ order statistic obtained from a sample of size $(n-r)$ from a population whose distribution is truncated on the left at x.

Result 2.3 (David and Nagaraja, 2003): Let $X_1, X_2, ..., X_n$ be a random sample from an absolutely continuous population with *df* $F(x)$ and *pdf* $f(x)$, and let $X_{1:n} \leq X_{2:n} \leq ... \leq X_{n:n}$ denote the order statistics obtained from this sample. Then the conditional distribution of $X_{s:n}$ given that $X_{r:n} = x$ and $X_{k:n} = z$ for $1 \leq r < s < k \leq n$, is the same as the distribution of the $(s-r)^{th}$ order statistic obtained from a sample of size $(k-r-1)$ from a population whose distribution is truncated on the left at x and on the right at z.

Remark 2.1: Result 2.1 follows from Result 2.3 by replacing k with $n+1$ with the convention $z = X_{n+1:n} = \beta$, where β is the upper range of X and $F(\beta) = 1$.

Remark 2.2: Result 2.2 follows from Result 2.3 by letting $r = 0$ the convention $X_{0:n} = x = \alpha$ (lower limit).

Remark 2.3: Order statistics in a sample from a continuous distribution form a Markov chain, that is

$$f(X_{k:n} | X_{1:n} = x_1, ..., X_{r:n} = x_r, ... X_{s:n} = x_s, ... X_{n:n} = x_n)$$
$$= f(X_{k:n} | X_{r:n} = x_r, X_{s:n} = x_s)$$

So, because of the Markovian properties of order statistics, it is of no use to condition it on more than two order statistics.

3. Sequential order statistics (Kamps, 1995)

In reliability theory, order statistics is used as a model to describe the reliability of k out of n systems. A k out of n systems consists of n components which start working simultaneously. It works while k components are functioning and breaks down if $n-k+1$ or more components fails. Parallel and series systems are particular cases of k out of n systems corresponding to the values $k=1$ and $k=n$ respectively. If a system consists of n components of the same kind and without any interactions with respect to life length distributions, are working then the system failure is modeled by an order statistics based on *iid rv*, assuming that the failure of one component does not effect the remaining components. However, in real life problems, generally this does not happen. The failures of some components can influence the remaining components. This can be thought of as damage caused by the i^{th} failure system. In this model, the life distribution of remaining components in the system may change after each failure of the components.

If we observe i^{th} failure at time x, the remaining components are now supposed to have a possibly different life length distribution. The distribution is truncated on the left of x to ensure realizations arranged in ascending order of magnitude.

Definition: Let $(Y_j^{(i)})_{1\le j\le n-i+1,\, 1\le i\le n}$ be independent *rv* with

$$(Y_j^{(i)})_{1\le j\le n-i+1} \sim F_i,\ 1\le i\le n,$$

where $F_1, F_2, \cdots, F_n$ are strictly increasing and continuous distribution functions and

$$F_1^{-1}(1)\le\cdots\le F_n^{-1}(1).$$

Moreover, let $X_j^{(1)}=Y_j^{(1)},\ 1\le j\le n,$

$$X_*^{(1)}=\min\{X_1^{(1)},\cdots,X_n^{(1)}\}$$

and for $2\le i\le n$

$$X_j^{(i)} = F_i^{-1}[F_i(Y_j^{(i)})(1-F_i(X_*^{(i-1)})) + F_i(X_*^{(i-1)})]$$

$$X_*^{(i)} = \min\{X_j^{(i)}, 1 \le j \le n-i+1\}$$

then the *rv* $X_*^{(1)}, \cdots, X_*^{(n)}$ are called sequential order statistics.

If we have absolutely continuous distribution functions $F_1, \cdots, F_n$, with densities $f_1, \cdots, f_n$, respectively, then the joint *pdf* of the r sequential order statistics $X_*^{(1)}, \cdots, X_*^{(r)}$ is given by

$$f_{X_*^{(1)}, \cdots, X_*^{(r)}}(x_1, \ldots, x_r) = \frac{n!}{(n-r)!} \prod_{i=1}^{r} \left[\frac{1-F_i(x_i)}{1-F_i(x_{i-1})}\right]^{n-i} \frac{f_i(x_i)}{[1-F_i(x_{i-1})]} \quad (3.1)$$

$$r \le n, \; x_0 = -\infty.$$

Sequential order statistics form a Markov chain with transition probabilities

$$P\left(X_*^{(r)} > t \,\middle|\, X_*^{(r-1)} = x\right) = \left(\frac{1-F_r(t)}{1-F_r(x)}\right)^{n-r+1}, \quad 2 \le r \le n. \quad (3.2)$$

4. Record values (Chandler, 1952)

In daily life we are often interested in observing new records and in recording them. Record values are defined by Chandler (1952) as a model for successive extremes in a sequence of *iid rv*. It may also be helpful as a model for successively largest insurance claims in non life insurance, for highest water levels or highest temperatures. Record values are also used in the reliability theory.

Definition: Let (X_i), $i \in N$ be a sequence of *iid* continuous *rv* with distribution function *df* $F(x)$ and *pdf* $f(x)$. Denote upper record times by $u(1) = 1$ and

$$u(n) = \min\{k > u(n-1) : X_k > X_{u(n-1)}\}.$$

The record value sequence is then defined by $X_{u(n)}, n = 1,2,\ldots$.

Based on an *iid* sequence of *rv* X_i, $i \in N$, with an absolutely continuous *df* $F(x)$ and *pdf* $f(x)$, the joint *pdf* of first r record values $X_{u(1)},\ldots,X_{u(r)}$ is given by (Ahsanullah, 1995)

$$f_{X_{u(1)},\cdots,X_{u(r)}}(x_1,\ldots,x_r) = \left(\prod_{i=1}^{r-1} \frac{f(x_i)}{1-F(x_i)}\right) f(x_r), \tag{4.1}$$

The marginal *pdf* of $X_{u(r)}$ is

$$f_{X_{u(r)}}(x) = \frac{1}{(r-1)!}[-\log \overline{F}(x)]^{r-1} f(x) \tag{4.2}$$

and the marginal *df* of $X_{u(r)}$ is

$$F_{X_{u(r)}}(x) = 1-[1-F(x)] \sum_{j=0}^{r-1} \frac{1}{j!}\left(\log \frac{1}{1-F(x)}\right)^j \tag{4.3}$$

The joint *pdf* of $X_{u(r)}$ and $X_{u(s)}$, $1 \le r < s$, is given by

$$f_{X_{u(r)},X_{u(s)}}(x,y) = \frac{1}{(r-1)!(s-r-1)!}[-\log \overline{F}(x)]^{r-1}$$

$$\times[-\log \overline{F}(y) + \log \overline{F}(x)]^{s-r-1} \frac{f(x)}{\overline{F}(x)} f(y), \quad \alpha < x < y < \beta \tag{4.4}$$

The joint *pdf* of $X_{u(r)}$, $X_{u(j)}$ and $X_{u(s)}$, $1 \le r < j < s$, can similarly be given as

$$f_{X_{u(r)},X_{u(j)},X_{u(s)}}(x,t,y) = \frac{1}{(r-1)!(j-r-1)!(s-j-1)!}[-\log \overline{F}(x)]^{r-1}$$

$$\times[-\log \overline{F}(t) + \log \overline{F}(x)]^{j-r-1}[-\log \overline{F}(y) + \log \overline{F}(t)]^{s-j-1} \frac{f(x)}{[\overline{F}(x)]} \frac{f(t)}{[\overline{F}(t)]} f(y)$$

$$\alpha < x < t < y < \beta, \tag{4.5}$$

where $\overline{F}(x) = 1 - F(x)$.

Hence the conditional *pdf* of $X_{u(j)}$ given $X_{u(r)} = x$ and $X_{u(s)} = y$, $1 \le r < j < s$ is

$$f_{X_{u(j)} \mid X_{u(r)}, X_{u(s)}}(t \mid x, y) = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!}$$

$$\times \frac{[-\log \overline{F}(t) + \log \overline{F}(x)]^{j-r-1} [-\log \overline{F}(y) + \log \overline{F}(t)]^{s-j-1}}{[-\log \overline{F}(y) + \log \overline{F}(x)]^{s-r-1}} \frac{f(t)}{[\overline{F}(t)]}$$

$$\alpha < x < t < y < \beta. \quad (4.6)$$

5. k-Records (Dziubdziela and Kopociński, 1976)

Consider the situation in which the record values themselves are viewed as outliers and hence second or third largest values are of special interest. Observing successive k^{th} largest value in a sequence, Dziubdziela and Kopociński (1976) proposed the following model of k^{th} record values.

Definition: Let (X_i), $i \in N$ be *iid rv* with a *df* $F(x)$ and let k be a positive integer. The random variable $u_{(k)}(n)$ given by

$$u_{(k)}(1) = 1,$$

$$u_{(k)}(n+1) = \min\{ j \in N; X_{j:j+k-1} > X_{u_{(k)}(n):\, u_{(k)}(n)+k-1} \},\ n \in N$$

are called k^{th} record times and the quantities $X_{u_{(k)}(n):\, u_{(k)}(n)+k-1}$, which we denote by $X_{u_{(k)}(n)}$, $n \in N$ are termed as k^{th} record values. Obviously, we obtain ordinary record values in the case $k = 1$.

Moreover, Nagaraja (1988) points out that k-records with an underlying *df* $F(x)$ can be viewed as ordinary record values $(k=1)$ based on the *df* $G(x)$ (minimum distribution) with

$$G(x)=1-(1-F(x))^k$$

Based on a sequence (X_i), $i\in N$, of *iid rv* possessing an absolutely continuous *df* $F(x)$ and *pdf* $f(x)$, the joint *pdf* of k-records $X_{u_{(k)}(1)},\dots,X_{u_{(k)}(r)}$

$$f_{X_{u_{(k)}(1)},\cdots,X_{u_{(k)}(r)}}(x_1,\dots,x_r)=k^r\left(\prod_{i=1}^{r-1}\frac{f(x_i)}{1-F(x_i)}\right)[1-F(x_r)]^{k-1}f(x_r), \tag{5.1}$$

The marginal *pdf* of $X_{u_{(k)}(r)}$ is

$$f_{X_{u_{(k)}(r)}}(x)=\frac{k^r}{(r-1)!}[-\log\bar{F}(x)]^{r-1}\,[1-F(x)]^{k-1}f(x) \tag{5.2}$$

and the marginal *df* of $X_{u_{(k)}(r)}$ is

$$F_{X_{u_{(k)}(r)}}(x)=1-[1-F(x)]^k\sum_{j=0}^{r-1}\frac{1}{j!}\left(k\log\frac{1}{\{1-F(x)\}}\right)^j \tag{5.3}$$

6. Progressive Type-II right censoring (Balakrishnan and Aggrawala, 2000)

Progressive censoring is used in life testing. In this scheme units can be removed at various stages during the experiment which may lead to saving of cost and time (Cohen, 1963; Sen, 1986). Under this scheme of censoring, from a total of n units placed on a life test, only m are completely observed until failure. At the time of the first failure, R_1 of the $n-1$ surviving units are randomly withdrawn (or censored) from the life testing experiment. At the time of the next failure, R_2 of $n-2-R_1$ surviving units are censored and so on. Finally at the time of the m^{th} failure, all the remaining $R_m = n - m - R_1 - \cdots - R_{m-1}$ surviving units are censored.

Suppose n *iid* units are placed on a life test with the corresponding failure times $X_1, X_2, \cdots, X_n$ being identically distributed with a continuous *df* $F(x)$ and *pdf* $f(x)$. Suppose further that the prefixed number of failures to be observed is m and that the progressive type-II right censoring scheme is $(R_1, R_2, \cdots, R_m)$. Then we shall denote the m completely observed failure times by

$X_{1:m:n}, X_{2:m:n}, \ldots, X_{m:m:n}$, where $X_{i:m:n}$, $i = 1, 2, \cdots, m$ denotes the i^{th} failure time in the progressive type-II right censoring scheme $(R_1, R_2, \cdots, R_m)$, bearing in mind that these still depend on the particular choice of $(R_1, R_2, \cdots, R_m)$ used. In a particular case when $R_1 = R_2 = \cdots = R_m = 0$, i.e. no withdrawals are carried out, then this model reduces to ordinary type-II censoring.

Above model is described as progressively Type-II right-censored order statistics from $F(x)$ arising from a sample of size n with the censoring scheme $(R_1, R_2, \cdots, R_m)$.

The *pdf* of all m progressively Type-II right-censored order statistics is

$$f_{X_{1:m:n}, \cdots, X_{m:m:n}}(x_1, \cdots, x_m) = C \prod_{i=1}^{m} f(x_i)[1 - F(x_i)]^{R_i} \qquad (6.1)$$

$$x_1 < x_2 < \cdots < x_n,$$

where

$$C = n(n - R_1 - 1) \cdots (n - R_1 - R_2 - \cdots - R_{m-1} - m + 1).$$

7. Generalized order statistics (Kamps, 1995)

Kamps introduced the model of generalized order statistics as follows.

Definition: Let $n \in N$, $k \geq 1$, $m_1, m_2, \cdots, m_{n-1} \in \Re^{n-1}$, $M_r = \prod_{j=r}^{n-1} m_j$, $1 \leq r \leq n-1$, be parameters such that $\gamma_r = k + n - r + M_r \geq 1$ for all $r \in \{1, \cdots, n-1\}$, and let $\tilde{m} = (m_1, \cdots, m_{n-1})$ and if $n \geq 2$, then $X(r, n, \tilde{m}, k)$, $r = 1, 2, \ldots, n$ are called generalized order statistics (*gos*) if their joint *pdf* is given by

$$f_{X(1,n,\tilde{m},k),\cdots,X(n,n,\tilde{m},k)}(x_1, \cdots, x_n) = k\left(\prod_{j=1}^{n-1} \gamma_j\right)\left(\prod_{i=1}^{n-1}(1 - F(x_i))^{m_i} f(x_i)\right)$$

$$\times (1 - F(x_n))^{k-1} f(x_n) \qquad (7.1)$$

on the cone

$$F^{-1}(0) < x_1 \leq \cdots \leq x_n < F^{-1}(1)$$

We will consider two cases.

Case I. In case I we will assume $m_1 = \cdots = m_{n-1} = m$ and write *gos* as $X(r, n, m, k)$, $r \geq 1$. Then in this case the marginal density function of the r^{th} *gos* based on an absolutely continuous *df* $F(x)$ with *pdf* $f(x)$ is given by

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!}(\bar{F}(x))^{k+n-r+M_r-1}\, g_m^{r-1}(F(x))\, f(x), \quad \alpha < x < \beta \qquad (7.2)$$

And the joint *pdf* of the *gos* $X(r,n,m,k)$ and $X(s,n,m,k)$, where $1 \le r < s \le n$, based on an absolutely continuous *df* $F(x)$ with *pdf* $f(x)$ is given by

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = C_{r,s:n}\ [\overline{F}(x)]^m g_m^{r-1}(F(x))$$
$$\times [h_m(F(y)) - h_m(F(x)]^{s-r-1} [\overline{F}(y)]^{k+(n-s)(m+1)-1} f(x) f(y)$$
$$\alpha < x < y < \beta \qquad (7.3)$$

The joint *pdf* of $X(r,n,m,k)$, $X(j,n,m,k)$ and $X(s,n,m,k)$, $1 \le r < j < s \le n$, can similarly be given as

$$f_{X(r,n,m,k),X(j,n,m,k),X(s,n,m,k)}(x,t,y)$$
$$= C_{r,j,s:n} [\overline{F}(x)]^m g_m^{r-1}(F(x)) [h_m(F(t)) - h_m(F(x)]^{j-r-1}$$
$$\times [h_m(F(y)) - h_m(F(t)]^{s-j-1} [\overline{F}(t)]^m [\overline{F}(y)]^{\gamma_s - 1} f(x) f(t) f(y)$$

$$\alpha < x < t < y < \beta \qquad (7.4)$$

Where $\overline{F}(x) = 1 - F(x)$,

$$C_{r,\ s:n} = \frac{c_{s-1}}{(r-1)!\ (s-r-1)!},$$

$$C_{r,j,s:n} = \frac{c_{s-1}}{(r-1)!(j-r-1)!(s-j-1)!},$$

$$c_{r-1} = \prod_{i=1}^{r} \gamma_i \qquad (7.5)$$

$$g_m(x) = h_m(x) - h_m(0) \qquad (7.6)$$

$$= \int_0^x (1-t)^m\, dt$$

$$\text{and} \qquad h_m(x) = \begin{cases} -\dfrac{(1-x)^{m+1}}{m+1} & m \neq -1 \\ \log\dfrac{1}{(1-x)} & m = -1 \end{cases} \quad , \; x \in (0,1) \qquad (7.7)$$

.

Therefore conditional distribution of $X(j, n, m, k)$ given $X(r, n, m, k) = x$ and $X(s, n, m, k) = y$ is given by

$$f_{(X(j,\,n,\,m,\,k)\ |\ X(r,\,n,\,m,\,k)=x,\, X(s,\,n,\,m,\,k)=y)}(t|x,y) = \frac{(s-r-1)!\,(m+1)}{(j-r-1)!(s-j-1)!}$$

$$\times \frac{[\{\overline{F}(x)\}^{m+1} - \{\overline{F}(t)\}^{m+1}]^{j-r-1}\,[\{\overline{F}(t)\}^{m+1} - \{\overline{F}(y)\}^{m+1}]^{s-j-1}}{[\{\overline{F}(x)\}^{m+1} - \{\overline{F}(y)\}^{m+1}]^{s-r-1}}[\overline{F}(t)]^{m} f(t)$$

$$\alpha < x < t < y < \beta \qquad (7.8)$$

Case II. For Case II, when $\gamma_i \neq \gamma_j$, $i \neq j$, the *pdf* of $X(r,n,\tilde{m},k)$ is given by (Kamps and Cramer, 2001)

$$f_{X(r,n,\tilde{m},k)}(x) = c_{r-1} \sum_{i=1}^{r} a_i(r)[\overline{F}(x)]^{\gamma_i - 1} f(x) \qquad (7.9)$$

and the joint *pdf* of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$, $1 \leq r < s \leq n$, is given by (Kamps and Cramer, 2001)

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y)$$

$$= c_{s-1}\left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left\{\frac{\overline{F}(y)}{\overline{F}(x)}\right\}^{\gamma_i}\right]\left[\sum_{i=1}^{r} a_i(r)(\overline{F}(x))^{\gamma_i}\right]\frac{f(x)}{\overline{F}(x)}\frac{f(y)}{\overline{F}(y)}$$

$$\alpha < x < \beta \qquad (7.10)$$

The joint *pdf* of $X(r, n, \tilde{m}, k)$, $X(j, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$, $1 \leq r < j < s \leq n$, may similarly be given as

$$f_{X(r,n,\tilde{m},k),X(j,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,t,y)$$

$$=c_{s-1}\left[\sum_{i=1}^{r} a_i(r)\,(\bar{F}(x))^{\gamma_i}\right]\left[\sum_{i=r+1}^{j} a_i^{(r)}(j)\left\{\frac{\bar{F}(t)}{\bar{F}(x)}\right\}^{\gamma_i}\right]$$

$$\times\left[\sum_{i=j+1}^{s} a_i^{(j)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(t)}\right\}^{\gamma_i}\right]\frac{f(x)}{\bar{F}(x)}\frac{f(t)}{\bar{F}(t)}\frac{f(y)}{\bar{F}(y)} \quad \alpha<x<y<\beta$$

(7.11)

Hence the conditional *pdf* of $X(j,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k)=x$ and $X(s,n,\tilde{m},k)=y$, $1\le r<j<s\le n$, is given by

$$f_{(X(j,\,n,\,\tilde{m},\,k)\;|\;X(r,\,n,\,\tilde{m},\,k)=x,\,X(s,\,n,\,\tilde{m},\,k)=y)}(t\,|\,x,y)$$

$$=\frac{\left[\sum\limits_{i=r+1}^{j} a_i^{(r)}(j)\left\{\frac{\bar{F}(t)}{\bar{F}(x)}\right\}^{\gamma_i}\right]\left[\sum\limits_{i=j+1}^{s} a_i^{(j)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(t)}\right\}^{\gamma_i}\right]\frac{f(t)}{\bar{F}(t)}}{\left[\sum\limits_{i=r+1}^{s} a_i^{(r)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(x)}\right\}^{\gamma_i}\right]},$$

$$\alpha<x<t<y<\beta, \quad (7.12)$$

where

$$a_i(r)=\prod_{\substack{j=1\\ j\ne i}}^{r}\frac{1}{(\gamma_j-\gamma_i)}, \qquad \gamma_j\ne\gamma_i \qquad 1\le i\le r\le n,$$

$$a_i^{(r)}(s)=\prod_{\substack{j=r+1\\ j\ne i}}^{s}\frac{1}{(\gamma_j-\gamma_i)}, \qquad \gamma_j\ne\gamma_i \qquad r+1\le i\le s\le n.$$

Generalized order statistics includes all the models related to ordered random variables. For instance it includes Ordinary order statistics [David and Nagaraja, 2003; Lawless, 1982; Arnold *et al.*, 1992; Balakrishnan and Rao, 1998 a, b], order statistics with non integral sample size [Stigler, 1977; Rohatgi and Saleh, 1988], records [Arnold *et al.*, 1998; Nevzorov, 2001], k^{th} records (Dziubdziela and Kopociński, 1976), sequential order statistics [Kamps, 1995; Cramer and Kamps, 2003], Progressive Type-II right censoring (Balakrishnan and Aggrawala, 2000).

Table 1.1: Variants of the generalized order statistics

		$\gamma_n = k$	γ_r	m_r
i)	Sequential order statistics	α_n	$(n-r+1)\alpha_r$	$(\gamma_r - \gamma_{r+1} - 1)$
ii)	Ordinary order statistics	1	$n-r+1$	0
iii)	Record values	1	1	-1
iv)	Progressively type II censored order statistics	$R_n + 1$	$n-r+1+\sum_{j=r}^{n} R_j$	R_r
v)	Pfeifer's record values	β_n	β_r	$(\beta_r - \beta_{r+1} - 1)$

8. Some special functions

1. Hypergeometric function: Hyper-geometric function is defined as (Mathai and Saxena, 1973)

$$ {}_pF_q[a_1,\cdots,a_p;b_1,\cdots,b_q;x]=\sum_{r=0}^{\infty}\left[\prod_{j=1}^{p}\frac{\Gamma(a_j+r)}{\Gamma(a_j)}\right]\left[\prod_{j=1}^{q}\frac{\Gamma(b_j)}{\Gamma(b_j+r)}\right]\frac{x^r}{r!} $$

The series ${}_pF_q[a_1,\cdots,a_p;b_1,\cdots,b_q;x]$ converges for $|x|<1$ diverges for $|x|>1$ and for $|x|=1$, ${}_pF_q$ converges only if $p=q+1$ and $\sum_{j=1}^{q}b_j-\sum_{j=1}^{p}a_j>0$ (Mathai and Saxena, 1973).

At $p=2$ and $q=1$

$$ {}_2F_1[a,b;c;x]=\sum_{r=0}^{\infty}\left[\frac{\Gamma(a+r)}{\Gamma(a)}\frac{\Gamma(b+r)}{\Gamma(b)}\right]\left[\frac{\Gamma(c)}{\Gamma(c+r)}\right]\frac{x^r}{r!} $$

is known as Guass hyper-geometric series,

and

$$ {}_3F_2[a,b,c;d,e;x]=\sum_{r=0}^{\infty}\left[\frac{\Gamma(a+r)}{\Gamma(a)}\frac{\Gamma(b+r)}{\Gamma(b)}\frac{\Gamma(c+r)}{\Gamma(c)}\right]\left[\frac{\Gamma(d)}{\Gamma(d+r)}\frac{\Gamma(e)}{\Gamma(e+r)}\right]\frac{x^r}{r!}. $$

Some important results in hyper-geometric functions are (Prudnikov *et al.*, 1986)

$$ {}_2F_1[a,b;c;1]=\frac{\Gamma(c)\,\Gamma(c-a-b)}{\Gamma(c-a)\,\Gamma(c-b)},\quad \mathrm{Re}(c-a-b)>0,\quad c\neq 0,-1,-2,\ldots $$

$$ \mathrm{B}_x(p,q)=\int_0^x u^{p-1}(1-u)^{q-1}du=p^{-1}x^p\,{}_2F_1(p,1-q;p+1;x), $$

and

$$\int_0^1 u^{a-1}(1-u)^{b-1}\,{}_2F_1(c,d;e;u)\,du = \mathrm{B}(a,b)\,{}_3F_2(c,d,a;e,a+b;1)$$

$$\int_0^1 u^{a-1}(1-u)^{b-1}\,{}_2F_1(c,d;a;u)du = \frac{\Gamma(a)\Gamma(b)\,\Gamma(a-c-d+b)}{\Gamma(a-c+b)\,\Gamma(a-d+b)}$$

$$[\mathrm{Re}\,a, \mathrm{Re}b, \mathrm{Re}(a+b-c-d)>0]$$

$$\int_0^1 u^{a-1}(1-u)^{b-1}\,{}_2F_1(c,a+b;e;u)du = \frac{\Gamma(e)\,\Gamma(a)\,\Gamma(b)\,\Gamma(e-c-a)}{\Gamma(e-c)\Gamma(e-a)\,\Gamma(a+b)}$$

$$[\,\mathrm{Re}\,a, \mathrm{Re}b, \mathrm{Re}(e-a-c)>0]$$

$$\int_0^1 u^{a-1}(1-u)^{d-e-n}\,{}_2F_1(-n,d;e;u)du = \frac{(e-a)_n}{(e)_n}\,\frac{\Gamma(a)\,\Gamma(d-e+1)}{\Gamma(e-a)\,\Gamma(d-e+a+1)}$$

$$[\,n=0,1,2\ldots;\ \mathrm{Re}\,a, \mathrm{Re}b>0]$$

$$\int_0^1 u^{c-e}(1-u)^{e-d-1}\,{}_2F_1(c,d;e;u)du = \frac{\sqrt{\pi}}{2^c}\,\frac{\Gamma(e)\,\Gamma(c-e+1)\,\Gamma(e-\frac{c}{2}-d)}{\Gamma(\frac{c+1}{2})\,\Gamma(e-\frac{c}{2})\,\Gamma(\frac{c}{2}-d+1)}$$

$$[\mathrm{Re}\,d<\mathrm{Re}\,e<\mathrm{Re}\ c+1;\ \mathrm{Re}(2e-2d-c)>0].$$

Formula for differentiation of definite integral (Prudnikov *et al.*, 1986)

$$\frac{\partial}{\partial x}\int_{u(x)}^{v(x)} A(x,t)\,dt = A[x, v(x)]\frac{\partial v(x)}{\partial x} - [x, u(x)]\frac{\partial u(x)}{\partial x} + \int_{u(x)}^{v(x)} \frac{\partial}{\partial x}A(x,t)\,dt$$

9. Some continuous distributions

I. Pareto distribution

A random variable X is said to have the Pareto distribution if its probability density function $(pdf)\, f(x)$ and distribution function $(df)\, F(x)$ are of the form given below:

$$f(x) = p\lambda^{p} x^{-(p+1)}; \qquad \lambda \le x < \infty; \quad \lambda, p > 0$$

$$F(x) = 1 - \lambda^{p} x^{-p}$$

Many socio-economic and naturally occurring quantities are distributed according to Pareto law. For example, distribution of city population sizes, personal income etc.

II. Power function distribution

A random variable X is said to have a power function distribution if its *pdf* and *df* are of the form given below:

$$f(x) = p\lambda^{-p} x^{p-1}; \qquad 0 \le x < \lambda; \quad \lambda, p > 0$$

$$F(x) = \lambda^{-p} x^{p}$$

The power function distribution is used to approximate representation of the lower tail of the distribution of random variable having fixed lower bound. It may be noted that if X has a power function distribution, then $Y = \dfrac{1}{X}$ has a Pareto distribution.

III. Beta distribution

a) Beta distribution of the first kind

A random variable X is said to have the beta distribution of the first kind if its *pdf* is of the form

$$f(x)=\frac{1}{B(p,q)}x^{p-1}(1-x)^{q-1};\ 0\leq x\leq 1;\ p,q>0$$

Beta distribution arises as the distribution of an ordered variable from a uniform distribution $U(0,1)$. Suppose $X_{r:n}$ is an ordered sample from $U(0,1)$, then $X_{r:n}$ is distributed as $B(r,n-r+1)$. The standard uniform distribution $U(0,1)$ is the special case of beta distribution of first kind obtained by putting the exponents p and q equal to 1. If $q=1$, the distribution reduces to power function distribution.

b) Beta distribution of the second kind

The continuous random variable X which is distributed according to probability law:

$$f(x)=\frac{1}{B(p,q)}\frac{x^{p-1}}{(1+x)^{p+q}};\ 0\leq x<\infty,\ p,q>0$$

is known as a beta variate of the second kind with parameters p and q.

Beta distribution of the second kind reduces to beta distribution of the first kind if we replace $1+x$ by $\frac{1}{y}$.

The beta distribution is one of the most frequently employed distributions to fit theoretical distributions. Beta distribution may be applied directly to the analysis of Markov processes with "uncertain" transition probabilities.

IV. Exponential distribution

A random variable X is said to have an exponential distribution if its *pdf* is given by

$$f(x) = \theta e^{-\theta x}; \qquad 0 \le x < \infty;\ \theta > 0,$$

and the *df* is given by

$$F(x) = 1 - e^{-\theta x}$$

The exponential distribution plays an important role in describing a large class of phenomena particularly in the area of reliability theory. The exponential distribution has many other applications. In fact, whenever a continuous random variable X assuming non-negative values satisfies the assumption,

$$P(X > s + t \mid X > s) = P(X > t) \text{ for all } s \text{ and } t,$$

then X will have an exponential distribution. This is particularly a very appropriate failure law when present does not depend on the past, for example, in studying the life of a bulb etc.

V. Rayleigh distribution

A random variable X is said to have a Rayleigh distribution if its *pdf* is given by:

$$f(x) = 2\theta\, x e^{-\theta x^2}; \qquad 0 \le x < \infty;\ \theta > 0,$$

and the *df* is given by

$$F(x) = 1 - e^{-\theta x^2}.$$

The Rayleigh distribution is derived by Lord Rayleigh. The Rayleigh distribution is quite appropriate for modeling the lifetimes of that component which possesses increasing failure rate.

VI. Weibull distribution

A random variable X is said to have a Weibull distribution if its *pdf* is given by:

$$f(x) = \theta\, p\, x^{p-1}\, e^{-\theta x^p}\,; \qquad 0 \le x < \infty;\ \theta, p > 0,$$

and the *df* is given by

$$F(x) = 1 - e^{-\theta x^p}$$

Weibull distribution is widely used in reliability and quality control. The distribution is also useful in cases where the conditions of strict randomness of exponential distribution are not satisfied. It is sometimes used as a tolerance distribution in the analysis of quantal response data.

If we put $p = 1$ in the Weibull distribution, we get the exponential distribution.

If we put $p = 2$, it gives Rayleigh distribution.

If X has a Weibull distribution, then the *pdf* of $Y = -p\log\left(\dfrac{X}{\alpha}\right)$ is

$$f(y) = e^{-y} e^{-e^{-y}}$$

which is a form of an extreme value distribution.

VII. Uniform distribution

A random variable X is said to have a uniform distribution or rectangular distribution if its *pdf* is given by

$$f(x)=\frac{1}{\lambda-\beta};\ \beta\leq x\leq\lambda,$$

and the *df* is given by

$$F(x)=\frac{x-\beta}{\lambda-\beta}$$

The standard uniform distribution $U(0,1)$ is obtained by putting $\beta=0$ and $\lambda=1$. It is noted that every distribution function $F(X)$ follows a uniform distribution $U(0,1)$. This distribution is used in "rounding off" errors, probability integral transformation, random number generation, traffic flow, generation of normal, exponential distribution etc.

VIII. Logistic distribution

A random variable X is said to have a Logistic distribution if its *pdf* is given by:

$$f(x)=\frac{\theta e^{-\theta x}}{(1+e^{-\theta x})^2};\qquad -\infty\leq x<\infty;\ \theta>0,$$

and the *df* is given by

$$F(x)=\frac{1}{(1+e^{-\theta x})}$$

Logistic distribution has several important applications in many different fields. For the application of Logistic distribution one may refer to Balakrishnan (1992).

IX. Burr distribution

Let X be a continuous random variable, then different forms of cumulative distribution function of X are listed below (Johnson *et al.*, 1994):

1 $F(x) = x; \quad 0 < x < 1$

2 $F(x) = (1 + e^{-x})^{-k}; \quad -\infty < x < \infty$

3 $F(x) = (1 + x^{-c})^{-k}; \quad 0 \le x < \infty$

4 $F(x) = \left[1 + \left(\frac{c - x}{x}\right)^{1/c}\right]^{-k}; \quad 0 \le x \le c$

5 $F(x) = [1 + ce^{-\tan x}]^{-k}; \quad -\frac{\pi}{2} \le x \le \frac{\pi}{2}$

6 $F(x) = [1 + ce^{-k \sinh x}]^{-k}; \quad -\infty < x < \infty$

7 $F(x) = 2^{-k}(1 + \tanh x)^{k}; \quad -\infty < x < \infty$

8 $F(x) = \left(\frac{2}{\pi} \tan^{-1} e^{x}\right)^{k}; \quad -\infty < x < \infty$

9 $F(x) = 1 - \frac{2}{c[(1 + e^{x})^{k} - 1] + 2}; \quad -\infty < x < \infty$

10 $F(x) = (1 - e^{-x^2})^{k}; \quad 0 \le x < \infty$

11 $F(x) = \left(x - \frac{1}{2\pi} \sin 2\pi x\right)^{k}; \quad 0 \le x \le 1$

12 $F(x) = 1 - (1 + x^{c})^{-k}; \quad 0 \le x < \infty$

where k and c are positive parameters.

Special attention is given to type XII, whose *pdf* is given as:

$$f(x) = kcx^{c-1}(1 + x^{c})^{-(k+1)}; \; 0 \le x < \infty; \; k, c > 0$$

This distribution is frequently used for the purpose of graduation and in reliability theory. At $c=1$, it is called **Lomax distribution** whereas at $k=1$, it is known as **Log-logistic distribution**.

X. Cauchy distribution

The special form of the Pearson type VII distribution, with *pdf*

$$f(x)=\frac{1}{\pi}\frac{1}{\lambda[1+\{(x-\theta)/\lambda\}^2]}; \qquad -\infty<x<\infty;\ \lambda>0,\ -\infty<\theta<\infty$$

is called the Cauchy distribution.

The *df* is given by

$$F(x)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}\left(\frac{x-\theta}{\lambda}\right)$$

The distribution is symmetrical about $x=\theta$. The distribution does not possess finite moments of order greater than or equal to 1, and so does not possess a finite expected value or standard deviation. However, θ and λ are location and scale parameters, respectively, and may be regarded as being analogous to mean and standard deviation.

There is no standard form of the Cauchy distribution, as it is not possible to standardize without using (finite) values of mean and standard deviation, which does not exist in this case. However, a standard form is obtained by putting $\theta=0, \lambda=1$. The standard probability density function is given by

$$f(x)=\frac{1}{\pi}\frac{1}{(1+x^2)}; \qquad -\infty<x<\infty$$

and the standard cumulative distribution function is

$$F(x)=\frac{1}{2}+\frac{1}{\pi}\tan^{-1}x.$$

It may be noted that if X and Y are standard normal variate then their ratio is Cauchy variate and also t-distribution at 1 degree of freedom is Cauchy distribution.

XI. Exponentiated Weibull distribution

The exponentiated Weibull distributiobn was introduced by Mudholkar et al. (1995). Its properties have been studied in more detail by Mudholkar and Hutson (1996) and Nassar and Eissa (2003). These authors have presented useful applications of the distribution in the modeling of flood data and in reliability. The *pdf* and the *df* of a random variable X having the exponentiated Weibull distributiobn are given by

$$f(x) = \tau\,\theta\,\lambda^{\theta} x^{\theta-1} e^{-(\lambda x)^{\theta}} [1- \ e^{-(\lambda x)^{\theta}}]^{\tau-1}; \ x>0; \ \lambda,\theta,\tau>0,$$

$$F(x) = [1-e^{-(\lambda x)^{\theta}}]^{\tau}$$

XII. Extreme value distribution

A random variable X is said to have extreme value distribution of type I if its *pdf* is given by

$$f(x) = e^{x} \exp[-e^{x}] \ ; \quad -\infty < x < \infty$$

and the *df* by

$$F(x) = 1-\exp[-e^{x}]$$

A random variable X is said to have extreme value distribution of type II if its *pdf* is given by

$$f(x) = p\,\theta^{p}\,x^{-(p+1)} e^{-\left(\frac{\theta}{x}\right)^{p}} \ ; \ 0<x<\infty,$$

and the *df* by

$$F(x) = e^{-\left(\frac{\theta}{x}\right)^{p}},$$

and a random variable X is said to have extreme value distribution of type III if its *pdf* is given by

$$f(x) = \frac{p}{\theta^{p}} (-x)^{p-1} e^{-\left(\frac{-x}{\theta}\right)^{p}} \quad ; \quad -\infty < x < 0,$$

and the *df* by

$$F(x) = e^{-\left(\frac{-x}{\theta}\right)^{p}}$$

The extreme value distribution is applied very much in natural phenomenon such as rain fall, floods, wind gusts, and air pollution.

Chapter II

ON CHARACTERIZATION OF CONTINUOUS DISTRIBUTIONS CONDITIONED ON A PAIR OF NON-ADJACENT RECORDS

1. Introduction

Characterization of distributions through conditional expectation of record values have been considered using $E[h(X_{u(j)})\big|X_{u(r)}=x]=a\,h(x)+b$ and $E[h(X_{u(j)})\big|X_{u(s)}=y]=a_1\,h(y)+b_1$ for $1\le r<j<s$ among others by Nagaraja (1977, 1988), Franco and Ruiz (1996, 1997), Wesolowski and Ahsanullah (1997), López-Blázquez and Moreno-Rebollo (1997), Ahsanullah and Wesolowski (1998), Dembińska and Wesolowski (2000), Wu and Lee (2001), Raqab (2002), Athar *et al.* (2003), Ahsanullah and Raqab (2004), Gupta and Ahsanullah (2004), Wu (2004) and Khan and Athar (2008). Bairamov *et al.* (2005) have characterized the exponential distribution and its monotone transforms, conditioned on a pair of adjacent record values. Here, in this chapter we have extended the results of Bairamov *et al.* (2005) and have characterized a family of continuous distributions through conditional expectation of record values based on a pair of non-adjacent records. The approach here is entirely different as given in Yanev *et al.* (2008).

2. Characterization of probability distributions

Conditional *pdf* of $X_{u(j)}$ given $X_{u(r)} = x$ and $X_{u(s)} = y$, $1 \le r < j < s$ is

$$f_{X_{u(j)} \mid X_{u(r)}, X_{u(s)}}(t \mid x, y) = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!}$$
$$\times \frac{[-\log \bar{F}(t) + \log \bar{F}(x)]^{j-r-1} [-\log \bar{F}(y) + \log \bar{F}(t)]^{s-j-1}}{[-\log \bar{F}(y) + \log \bar{F}(x)]^{s-r-1}} \frac{f(t)}{[\bar{F}(t)]}$$
$$\alpha < x < t < y < \beta \quad (2.1)$$

Theorem 2.1: Let $X_{u(1)}, X_{u(2)}, \ldots$ be the upper records from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α, β) and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values r and $r+1$, $1 \le r < j-1 < s$,

$$g_{j \mid l, s}(x, y) = E[h(X_{u(j)}) \mid X_{u(l)} = x, X_{u(s)} = y] \ , \quad l = r, r+1, \qquad (2.2)$$

exist, where $g(\ )$ is a finite and differentiable function of x,

then

$$\frac{\log \bar{F}(x)}{\log \bar{F}(y)} = 1 - e^{-I_1}, \qquad (2.3)$$

where

$$A_1(x, y) = \frac{\frac{\partial}{\partial x} g_{j \mid r, s}(x, y)}{(s-r-1)\,[g_{j \mid r, s}(x, y) - g_{j \mid r+1, s}(x, y)]},$$

and $\quad I_1 = \int_{\alpha}^{x} A_1(t, y)\, dt$

Proof: We have,

$$g_{j|r,s}(x,y) = E[h(X_{u(j)}) \,|\, X_{u(r)} = x, X_{u(s)} = y]$$

Therefore in view of (2.1),

$$g_{j|r,s}(x,y)[B(x,y)]^{s-r-1} = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!}\int_x^y h(t)[B(x,t)]^{j-r-1}[B(t,y)]^{s-j-1}\frac{f(t)}{[\bar{F}(t)]}dt \tag{2.4}$$

where

$$B(x,y) = [-\log\bar{F}(y) + \log\bar{F}(x)] \tag{2.5}$$

Differentiating both the sides $w.r.t.$ x, we have

$$\frac{\partial}{\partial x} g_{j|r,s}(x,y)\,[B(x,y)]^{s-r-1} - g_{j|r,s}(x,y)\,(s-r-1)\frac{f(x)}{[\bar{F}(x)]}[B(x,y)]^{s-r-2}$$

$$= -\frac{(s-r-1)!}{(j-r-2)!(s-j-1)!}\frac{f(x)}{[\bar{F}(x)]}\int_x^y h(t)\,[B(x,t)]^{j-r-2}\,[B(t,y)]^{s-j-1}\frac{f(t)}{[\bar{F}(t)]}\,dt$$

$$= -(s-r-1)\frac{f(x)}{[\bar{F}(x)]}\,[B(x,y)]^{s-r-2}\,g_{j|r+1,s}(x,y),$$

implying that,

$$\frac{f(x)}{[\bar{F}(x)]\,B(x,y)} = \frac{\frac{\partial}{\partial x} g_{j|r,s}(x,y)}{(s-r-1)\,[g_{j|r,s}(x,y) - g_{j|r+1,s}(x,y)]} = A_1(x,y)$$

Integrating both the sides $w.r.t.$ x over (α, x), we get

$$\log B(x,y)\Big|_\alpha^x = -\int_\alpha^x A_1(t,y)\,dt$$

and hence the result.

Remark 2.1: From (2.3), we have

$$\left[1+\frac{\log\bar{F}(x)}{-\log\bar{F}(y)}\right]^{s-r-1}=\exp\left[-\int_{\alpha}^{x}A(t,y)\,dt\right], \tag{2.6}$$

where $A_1(x,y)=\dfrac{A(x,y)}{(s-r-1)}$.

As $y\to\beta$, $-\log\bar{F}(y)\to\infty$, therefore in the limiting case as $s\to\infty$ and $y\to\beta$, L.H.S. of (2.6) tends to $\bar{F}(x)$, implying that

$$\bar{F}(x)=\exp\left[-\int_{\alpha}^{x}A(t)\,dt\right],$$

where,

$$A(x)=\frac{\frac{\partial}{\partial x}g_{j|\,r}(x)}{[g_{j|\,r}(x)-g_{j|\,r+1}(x)]},$$

and $g_{j|\,r}(x)=E[h(X_{u(j)})\,|\,X_{u(r)}=x]$,

as obtained by Khan and Athar (2009).

Theorem 2.2: Let $X_{u(1)},X_{u(2)},\ldots$ be the upper records from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α,β) and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values $s-1$ and s, $1\le r<j+1<s$,

$$g_{j|\,r,l}(x,y)=E[h(X_{u(j)})\,|\,X_{u(r)}=x,X_{u(l)}=y]\ ,\ l=s,s-1, \tag{2.7}$$

exist, where $g(\ )$ is a finite and differentiable function of y,

then,

$$\frac{1+\log\bar{F}(y)}{1+\log\bar{F}(x)}=1-e^{-I_2}\ ,\ x<q, \tag{2.8}$$

where $q \in (\alpha, \beta)$ such that $-\log \bar{F}(q) = 1$,

$$A_2(x, y) = \frac{\frac{\partial}{\partial y} g_{j|\, r,s}(x, y)}{(s-r-1)\,[g_{j|\, r,s}(x, y) - g_{j|\, r,s-1}(x, y)]},$$

and $\quad I_2 = \int_q^y A_2(x,t)\, dt.$

Proof: We have,

$$g_{j|\, r,s}(x, y)[B(x, y)]^{s-r-1} = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!} \int_x^y h(t)[B(x,t)]^{j-r-1}[B(t, y)]^{s-j-1} \frac{f(t)}{[\bar{F}(t)]} dt$$

Differentiating both the sides *w.r.t.* y, we have

$$\frac{\partial}{\partial y} g_{j|\, r,s}(x, y)[B(x, y)]^{s-r-1} + g_{j|\, r,s}(x, y)(s-r-1)\frac{f(y)}{[\bar{F}(y)]}[B(x, y)]^{s-r-2}$$

$$= \frac{(s-r-1)!}{(j-r-1)!(s-j-2)!} \frac{f(y)}{[\bar{F}(y)]} \int_x^y h(t)[B(x,t)]^{j-r-1}[B(t, y)]^{s-j-2} \frac{f(t)}{[\bar{F}(t)]} dt$$

$$= (s-r-1)\frac{f(y)}{[\bar{F}(y)]}[B(x, y)]^{s-r-2}\, g_{j|\, r,s-1}(x, y)$$

implying that,

$$\frac{f(y)}{[\bar{F}(y)]\, B(x, y)} = -\frac{\frac{\partial}{\partial y} g_{j|\, r,s}(x, y)}{(s-r-1)\,[g_{j|\, r,s}(x, y) - g_{j|\, r,s-1}(x, y)]} = -A_2(x, y)$$

Integrating both the sides *w.r.t.* y over (q, y), we have

$$\log B(x, y)\Big|_q^y = -\int_q^y A_2(x,t)\, dt,$$

and hence the Theorem.

Remark 2.2: By convention $X_{u(0)} = \alpha$, therefore we have:

If $g_{j|s}(y) = E[h(X_{u(j)}) \mid X_{u(s)} = y]$

then

$$-\log \overline{F}(y) = \exp\left(- \int_q^y B(t)\, dt \right),$$

a result given by Khan and Athar (2009), conditioned on a single record statistic, where

$$B(t) = \frac{g'_{j|s}(t)}{(s-1)\,[g_{j|s}(t) - g_{j|s-1}(t)]}.$$

In Theorem 2.1 we have assumed that $g(x, y)$ is differentiable *w.r.t.* x, whereas in Theorem 2.2, $g(x, y)$ is assumed to be differentiable *w.r.t.* y. However if $g(x, y)$ is assumed to be differentiable *w.r.t.* both x and y, we can combine Theorem 2.1 and Theorem 2.2 as:

Theorem 2.3: Under the conditions of Theorem 2.1 and Theorem 2.2

$$\overline{F}(x) = \exp\left[- \frac{e^{I_1} - 1}{e^{I_1} + e^{I_2} - 1} \right], \tag{2.9}$$

and,

$$\overline{F}(y) = \exp\left[- \frac{e^{I_1}}{e^{I_1} + e^{I_2} - 1} \right] \tag{2.10}$$

Proof: In view of (2.3) and (2.8), we get

$$e^{I_1} = \frac{\log \overline{F}(y)}{\log \overline{F}(y) - \log \overline{F}(x)} \quad \text{and} \quad e^{I_1} + e^{I_2} - 1 = - \frac{1}{\log \overline{F}(y) - \log \overline{F}(x)},$$

and hence the Theorem.

Now we deduce Theorems for specific family of distributions.

Corollary 2.1:

$$g_{j|r,s}(x,y) = E[h(X_{u(j)}) \,|\, X_{u(r)} = x, X_{u(s)} = y] = \frac{(s-j)h(x) + (j-r)h(y)}{(s-r)} \tag{2.11}$$

if and only if the *df* is

$$F(x) = 1 - e^{-[ah(x)+b]} \quad , \quad \alpha \le x \le \beta, \tag{2.12}$$

when $h(t)$ is a non-decreasing and differentiable function of t for $a > 0$,

and the *df* is

$$G(x) = e^{-[ah(x)+b]} \quad , \quad \alpha \le x \le \beta \tag{2.13}$$

when $h(t)$ is a non-increasing and differentiable function of t for $a > 0$.

Proof: First we prove that (2.12) implies (2.11).

Let $C_{r,\,j,\,s} = \dfrac{(s-r-1)!}{(j-r-1)!(s-j-1)!}$,

then

$$g_{j|r,s}(x,y) = \frac{C_{r,\,j,\,s}}{[-\log\overline{F}(y) + \log\overline{F}(x)]}$$

$$\times \int_x^y h(t) \left[\frac{\{-\log\overline{F}(t) + \log\overline{F}(x)\}}{\{-\log\overline{F}(y) + \log\overline{F}(x)\}}\right]^{j-r-1} \left[1 - \frac{\{-\log\overline{F}(t) + \log\overline{F}(x)\}}{\{-\log\overline{F}(y) + \log\overline{F}(x)\}}\right]^{s-j-1} \frac{f(t)}{[\overline{F}(t)]} dt$$

$$= \frac{C_{r,\,j,\,s}}{[h(y) - h(x)]} \int_x^y h(t) \left[\frac{\{h(t) - h(x)\}}{\{h(y) - h(x)\}}\right]^{j-r-1} \left[1 - \frac{\{h(t) - h(x)\}}{\{h(y) - h(x)\}}\right]^{s-j-1} h'(t)\,dt$$

Set $u = \dfrac{h(t) - h(x)}{h(y) - h(x)}$ to get

$$g_{j|r,s}(x,y) = C_{r,j,s} \int_0^1 [h(x) + u\{h(y) - h(x)\}]\, u^{j-r-1} (1-u)^{s-j-1}\, du$$

$$= \frac{(s-j)\, h(x) + (j-r)\, h(y)}{(s-r)}$$

To prove (2.11) implies (2.12), note that

$$A_1(x,y) = \frac{\frac{\partial}{\partial x} g_{j|r,s}(x,y)}{(s-r-1)\,[g_{j|r,s}(x,y) - g_{j|r+1,s}(x,y)]} = \frac{h'(x)}{[h(y) - h(x)]}$$

Thus in view of the Theorem 2.1,

$$\frac{\log \overline{F}(x)}{\log \overline{F}(y)} = \frac{a\,h(x) + b}{a\,h(y) + b}$$

That is

$$\log \overline{F}(x) = K\,[ah(x) + b],$$

where K is a normalizing constant. If $h(x)$ is a non-decreasing then $0 < -\log \overline{F}(x) < \infty$, and therefore there exist a q such that $-\log \overline{F}(q) = 1$. Hence

$$F(x) = 1 - e^{-[ah(x)+b]}$$

Similarly for $h(x)$ non-increasing, $-\log G(q) = 1$, and hence

$$G(x) = e^{-[ah(x)+b]}.$$

Theorem 2.2 and Theorem 2.3 may also be used to prove the Corollary 2.1.

Table 2.1: Examples based on the distribution $F(x) = 1 - e^{-[a\,h(x)+b]}$

Distribution	$F(x)$	a	b	$h(x)$
Power function	$v^{-p}x^{p}$ $0 < x < v$	1	$p\log v$	$-\log(v^{p} - x^{p})$
Pareto	$1 - v^{p}x^{-p}$ $v < x < \infty$	p	$-p\log v$	$\log x$
Beta of the I kind	$1-(1-x)^{p}$ $0 < x < 1$	p	0	$-\log(1-x)$
Exponential	$1 - e^{-\theta x}$ $0 < x < \infty$	θ	0	x
Rayleigh	$1 - e^{-\theta x^{2}}$ $0 < x < \infty$	θ	0	x^{2}
Weibull	$1 - e^{-\theta x^{p}}$ $0 < x < \infty$	θ	0	x^{p}
Extreme value II	$1 - \exp[-e^{\theta x}]$ $-\infty < x < \infty$	1	0	$e^{\theta x}$
Burr Type XII	$1-(1+\theta x^{p})^{-m}$ $0 < x < \infty$	m	0	$\log(1+\theta x^{p})$

Table 2.2: Examples based on the distribution $G(x) = e^{-[a\,h(x)+b]}$

Distribution	$G(x)$	a	b	$h(x)$
Power function	$v^{-p}x^{p}$ $0 < x < v$	p	$p\log v$	$-\log x$
Inverse Weibull	$e^{-\theta x^{-p}}$ $0 < x < \infty$	θ	0	x^{-p}
Gumbel	$\exp[-c\,e^{-x}]$ $-\infty < x < \infty$	c	0	e^{-x}
Logistic	$(1+e^{-x})^{-c}$ $-\infty < x < \infty$	c	0	$\log(1+e^{-x})$

Remark 2.3: Bairamov *et al.* (2005) have given three examples when $h(x)$ is non-decreasing for adjacent records at $j = n$, $r = n-1$ and $s = n+1$. Their results are essentially as given in the Table 2.1 for Weibull, beta of the first kind and Pareto distributions. Their remaining two examples are for Gumbel and logistic distributions when $h(x)$ is non-increasing, as given in the Table 2.2.

3. Results for adjacent records

In this section, we have deduced the results for adjacent records from Section 2. Ruiz and Navarro (1996) have also characterized distributions when conditioned records are adjacent but our approach is entirely different.

To this end, we define,

$$g_{r|\,r,s}(x,y)=E[h(X_{u(r)})\,|\,X_{u(r)}=x,X_{u(s)}=y]=h(x) \tag{3.1}$$

$$g_{s|\,r,s}(x,y)=E[h(X_{u(s)})\,|\,X_{u(r)}=x,X_{u(s)}=y]=h(y) \tag{3.2}$$

and

$$g_{r+1|\,r,\,r+2}(x,y)=E[h(X)\,|\,x\leq X\leq y]=m(x,y) \tag{3.3}$$

Therefore at $j=r+1$ and $s=r+2$

$$A_1(x,y)=\frac{\frac{\partial}{\partial x}m(x,y)}{[m(x,y)-h(x)]} \tag{3.4}$$

and

$$A_2(x,y)=\frac{\frac{\partial}{\partial y}m(x,y)}{[h(y)-m(x,y)]} \tag{3.5}$$

Now, we deduce the characterizing results obtained in Section 2 for adjacent records in Corollary 3.1, 3.2 and 3.3.

Corollary 3.1:

$$m(x,y)=\frac{c}{a(c+1)}\,\frac{[ah(y)+b]^{c+1}-[ah(x)+b]^{c+1}}{[ah(y)+b]^{c}-[ah(x)+b]^{c}}-\frac{b}{a}, \qquad c\neq -1 \tag{3.6}$$

if and only if

$$-\log\bar{F}(x)=[ah(x)+b]^{c}, \tag{3.7}$$

where a, b, c and $h(x)$ are so chosen that $F(x)$ is a *df*.

Proof: For $-\log\bar{F}(x)=[ah(x)+b]^{c}$, we have

$$m(x,y)=\int_x^y \frac{h(t)f(t)}{[\bar{F}(t)][-\log\bar{F}(y)+\log\bar{F}(x)]}dt$$

$$= \frac{ac}{B(x,y)} \int_x^y [ah(t)+b]^{c-1} h(t)\, h'(t)\; dt \;,$$

where

$$B(x,y) = [-\log \bar{F}(y) + \log \bar{F}(x)] = [ah(y)+b]^c - [ah(x)+b]^c$$

That is,

$$m(x,y) = \frac{c}{B(x,y)} \int_{ah(x)+b}^{ah(y)+b} u^{c-1}\left(\frac{u-b}{a}\right) du$$

implying that,

$$m(x,y) = \frac{c}{a(c+1)} \frac{[ah(y)+b]^{c+1} - [ah(x)+b]^{c+1}}{[ah(y)+b]^c - [ah(x)+b]^c} - \frac{b}{a}, \quad c \neq -1$$

Now to prove (3.6) implies (3.7), we have

$$A_1(x,y) = \frac{\frac{\partial}{\partial x} m(x,y)}{[m(x,y) - h(x)]} = \frac{ac\, h'(x)\{ah(x)+b\}^{c-1}}{[\{ah(y)+b\}^c - \{ah(x)+b\}^c]}$$

Integrating both the sides *w.r.t.* x, we get

$$-\int_\alpha^x A_1(t,y)\, dt = \log\left[1 - \frac{\{ah(x)+b\}^c}{\{ah(y)+b\}^c}\right]$$

Therefore in view of the Theorem 2.1,

$$\frac{\log \bar{F}(x)}{\log \bar{F}(y)} = \frac{\{ah(x)+b\}^c}{\{ah(y)+b\}^c}$$

Thus

$$-\log \bar{F}(x) = [ah(x)+b]^c .$$

Corollary 3.2:

$$m(x,y)=\frac{c}{a(c+1)}\frac{[ah(x)+b]^{c+1}-[ah(y)+b]^{c+1}}{[ah(x)+b]^{c}-[ah(y)+b]^{c}}-\frac{b}{a}\ ,\quad c\neq -1 \tag{3.8}$$

if and only if

$$1-[-\log \bar{F}(x)]=[ah(x)+b]^{c}, \tag{3.9}$$

where a, b, c and $h(x)$ are such that $F(x)$ is a *df.*

Proof: For $1-[-\log \bar{F}(x)]=[ah(x)+b]^{c}$, we have

$$m(x,y)=\int_{x}^{y}\frac{h(t)f(t)}{[\bar{F}(t)][\{1-\log \bar{F}(y)\}-\{1-\log \bar{F}(x)\}]}dt$$

$$=\frac{ac}{B(x,y)}\int_{x}^{y}[ah(t)+b]^{c-1}h(t)h'(t)\,dt$$

where

$$B(x,y)=[\{1-\log \bar{F}(y)\}-\{1-\log \bar{F}(x)\}]=[ah(y)+b]^{c}-[ah(x)+b]^{c}$$

That is,

$$m(x,y)=\frac{c}{B(x,y)}\int_{ah(x)+b}^{ah(y)+b}u^{c-1}\left(\frac{u-b}{a}\right)du$$

$$=\frac{c}{a(c+1)}\frac{[ah(x)+b]^{c+1}-[ah(y)+b]^{c+1}}{[ah(x)+b]^{c}-[ah(y)+b]^{c}}-\frac{b}{a}.$$

Now to prove (3.8) implies (3.9), proceeding as in Corollary 3.1, we get

$$A_2(x,y)=\frac{\frac{\partial}{\partial y}m(x,y)}{[h(y)-m(x,y)]}=-\frac{ac\,h'(y)\{ah(y)+b\}^{c-1}}{[\{ah(x)+b\}^{c}-\{ah(y)+b\}^{c}]}$$

Integrating both the sides *w.r.t.* y, we get

$$\frac{1+\log \overline{F}(y)}{1+\log \overline{F}(x)} = \frac{\{ah(y)+b\}^c}{\{ah(x)+b\}^c}$$

Thus,

$$1-[-\log \overline{F}(y)] = [ah(x)+b]^c$$

Corollary 3.3: $m(x,y) = \frac{c}{(c-1)} \frac{[\{h(x)\}^{c-1} - \{h(y)\}^{c-1}]\, h(x)h(y)}{[h(x)]^c - [h(y)]^c}$, $c \neq 1$

(3.10)

if and only if

$$-\log \overline{F}(x) = a[h(x)]^{-c} + b, \tag{3.11}$$

where a, b, c and $h(x)$ are such that $F(x)$ is a *df.*

Proof: For $-\log \overline{F}(x) = a[h(x)]^{-c} + b$, we have,

$$m(x,y) = \int_x^y \frac{h(t)f(t)}{[\overline{F}(t)][-\log \overline{F}(y) + \log \overline{F}(x)]} dt$$

$$= -\frac{ac}{B(x,y)} \int_x^y [h(t)]^{-c-1} h(t)\, h'(t)\, dt$$

$$= \frac{c}{(c-1)} \frac{[\{h(x)\}^{c-1} - \{h(y)\}^{c-1}]\, h(x)h(y)}{[\{h(x)\}^c - \{h(y)\}^c]}$$

Now to prove (3.10) implies (3.11), we have

$$A_1(x,y) = \frac{\frac{\partial}{\partial x} m(x,y)}{[m(x,y) - h(x)]} = -\frac{c\,h'(x)\{h(y)\}^c}{h(x)[\{h(x)\}^c - \{h(y)\}^c]}$$

$$= -\frac{c\,h'(x)\{h(y)\}^{c}}{\{h(x)\}^{c+1}\left[1-\left\{\frac{h(y)}{h(x)}\right\}^{c}\right]}$$

Integrating both the sides *w.r.t.* x,

$$-\int_{\alpha}^{x} A_1(t,y)\,dt = \log\left[1-\frac{a\{h(x)\}^{-c}+b}{a\{h(y)\}^{-c}+b}\right]$$

we get,

$$-\log \overline{F}(x) = a[h(x)]^{-c}+b.$$

Remark 3.1: For $\overline{F}(x) = e^{-a[h(x)]^{-c}-b}$, we have

(a) At $c=-1$ and $\overline{F}(x) = e^{-a\,h(x)-b}$,

$$m(x,y) = \frac{1}{2}\left(\frac{1}{h(x)}+\frac{1}{h(y)}\right)h(x)\,h(y) = \frac{h(x)+h(y)}{2} = \text{A.M.} \qquad (3.12)$$

For $h(x)=x$, it is an exponential distribution, as also given in Table 2.1.

(b) At $c=2$ and $\overline{F}(x) = e^{-a[h(x)]^{-2}-b}$,

$$m(x,y) = \frac{2\,h(x)\,h(y)}{h(x)+h(y)} = \frac{1}{\frac{1}{2}\left(\frac{1}{h(x)}+\frac{1}{h(y)}\right)} = \text{H.M.} \qquad (3.13)$$

(c) At $c=\frac{1}{2}$ and $\overline{F}(x) = e^{-a[h(x)]^{-1/2}-b}$,

$$m(x,y) = \frac{\left(\frac{1}{\sqrt{h(x)}}-\frac{1}{\sqrt{h(y)}}\right)h(x)\,h(y)}{\sqrt{h(x)}-\sqrt{h(y)}} = \frac{(\sqrt{h(y)}-\sqrt{h(x)})\,h(x)h(y)}{\sqrt{h(x)h(y)}(\sqrt{h(y)}-\sqrt{h(x)})}$$

$$= \sqrt{h(x)h(y)} = \text{G.M} \qquad (3.14)$$

Where A.M., H.M. and G.M. are arithmetic mean, harmonic mean and geometric mean respectively. That is, these results specify the probability distributions if the conditional expectation is A.M., H.M. and G.M.

Chapter III

ON CHARACTERIZATION OF CONTINUOUS DISTRIBUTIONS CONDITIONED ON A PAIR OF GENERALIZED ORDER STATISICS

1. Introduction

Since Kamps (1995) introduced the concept of the *gos*, several characterizing results have appeared in the literature using conditional expectation. Keseling (1999), Bienik and Szynal (2003), Bienik (2007), Cramer *et al.* (2004), Khan and Alzaid (2004), Raqab and Abu-Lawi (2004), Ahsanullah and Raqab (2004), Beg and Ahsanullah (2006), Khan *et al.* (2006) and Samuel (2008) have characterized distributions based on the conditional expectation of *gos* conditioned on adjacent and non-adjacent *gos*. Ahsanullah and Beg (2008) have characterized continuous distributions through conditional expectation of *gos* conditioned on two adjacent *gos*.

We, in this chapter, have extended the result when regression based on two non-adjacent *gos* may not be linear. Then the result is deduced for known results on *gos*, records and order statistics.

Part of this chapter has appeared in Ahsanullah *et al.* (2009).

The concept of *gos* is introduced in Chapter I. We, in this chapter, will obtain the characterizing results for

(1) $m_1 = \cdots = m_{n-1} = m$ and

(2) $\gamma_j \neq \gamma_i$, $i \neq j$.

2. Characterization of distributions when $m_i = m_j, i, j = 1, \ldots, n-1$

Let $X(i, n, m, k)$, $i = 1, \ldots, n$ be the *gos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α, β), then the conditional *pdf* of $X(j, n, m, k)$ given $X(r, n, m, k) = x$ and $X(s, n, m, k) = y$, $1 \leq r < j < s \leq n$ is given by

$$f_{j|r,s}(t|x,y) = \frac{(s-r-1)!(m+1)}{(j-r-1)!(s-j-1)!}$$

$$\times \frac{[\{\bar{F}(x)\}^{m+1} - \{\bar{F}(t)\}^{m+1}]^{j-r-1}[\{\bar{F}(t)\}^{m+1} - \{\bar{F}(y)\}^{m+1}]^{s-j-1}}{[\{\bar{F}(x)\}^{m+1} - \{\bar{F}(y)\}^{m+1}]^{s-r-1}}[\bar{F}(t)]^m f(t)$$

$$\alpha \leq x \leq t \leq y \leq \beta \qquad (2.1)$$

Theorem 2.1: Let $X(i, n, m, k)$, $i = 1, \ldots, n$, be the *gos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α, β)s and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values r and $r+1$, $1 \leq r < j-1 < s$,

$$g_{j|l,s}(x,y) = E[h\{X(j,n,m,k)\} | X(l,n,m,k) = x, X(s,n,m,k) = y],\ l = r, r+1 \qquad (2.2)$$

exist, where $g(\,)$ is a finite and differentiable function of x,

then

$$\frac{1-\{\bar{F}(x)\}^{m+1}}{1-\{\bar{F}(y)\}^{m+1}} = 1 - e^{-I_1}, \qquad m \neq -1 \qquad (2.3)$$

$$\frac{\log \bar{F}(x)}{\log \bar{F}(y)} = 1 - e^{-I_1}, \qquad m = -1 \tag{2.4}$$

where

$$A_1(x,y) = \frac{\frac{\partial}{\partial x} g_{j|r,s}(x,y)}{(s-r-1)\,[g_{j|r,s}(x,y) - g_{j|r+1,s}(x,y)]}$$

and

$$I_1 = \int_{\alpha}^{x} A_1(t,y)\,dt \tag{2.5}$$

Proof: We have,

$$g_{j|r,s}(x,y) = E[h\{X(j,n,m,k)\} \mid X(r,n,m,k) = x,\ X(s,n,m,k) = y]$$

Let $B(x,y) = [\{\bar{F}(x)\}^{m+1} - \{\bar{F}(y)\}^{m+1}]$, (2.6)

then

$$g_{j|r,s}(x,y)[B(x,y)]^{s-r-1} = \frac{(s-r-1)!\,(m+1)}{(j-r-1)!(s-j-1)!}$$

$$\times \int_{x}^{y} h(t)[\bar{F}(t)]^{m}[B(x,t)]^{j-r-1}[B(t,y)]^{s-j-1} f(t)\,dt$$

Differentiating both the sides *w.r.t.* x, we have

$$\frac{\partial}{\partial x} g_{j|r,s}(x,y)[B(x,y)]^{s-r-1} - g_{j|r,s}(x,y)(s-r-1)(m+1)f(x)[\bar{F}(x)]^{m}[B(x,y)]^{s-r-2}$$

$$= -\frac{(s-r-1)!(m+1)^2}{(j-r-2)!(s-j-1)!} \int_{x}^{y} h(t)[\bar{F}(t)]^{m} f(x)[\bar{F}(x)]^{m}[B(x,t)]^{j-r-2}[B(t,y)]^{s-j-1} f(t)\,dt$$

$$\frac{\partial}{\partial x} g_{j|r,s}(x,y)[B(x,y)]^{s-r-1} - g_{j|r,s}(x,y)(s-r-1)(m+1)f(x)[\bar{F}(x)]^{m}[B(x,y)]^{s-r-2}$$

$$= -(s-r-1)(m+1)f(x)[\bar{F}(x)]^{m}[B(x,y)]^{s-r-2} g_{j|r+1,s}(x,y)$$

That is,

$$\frac{(m+1)f(x)[\overline{F}(x)]^m}{B(x,y)} = \frac{\frac{\partial}{\partial x} g_{j|r,s}(x,y)}{(s-r-1)\,[g_{j|r,s}(x,y) - g_{j|r+1,s}(x,y)]} = A_1(x,y)$$

Integrating both the sides *w.r.t.* x over (α, x), we have

$$\log B(x,y)\Big|_{\alpha}^{x} = -\int_{\alpha}^{x} A_1(t,y)\,dt,$$

implying that

$$\log\left[1 - \frac{1-\{\overline{F}(x)\}^{m+1}}{1-\{\overline{F}(y)\}^{m+1}}\right] = -\int_{\alpha}^{x} A_1(t,y)\,dt,$$

as $\log \overline{F}(\alpha) = 0$, and hence (2.3) is proved. If we take limit as $m \to -1$ in (2.3), we get (2.4) as given in Theorem 2.1 in Chapter II.

Remark 2.1: At $s = n + \frac{k}{m+1}$, by convention $X\left(n + \frac{k}{(m+1)}, n, m, k\right) = y$ $= \beta$, and hence $\overline{F}(y) = 0$.

Therefore,

$$g_{j|r}(x) = E[h(X(j,n,m,k)) \mid X(r,n,m,k) = x],$$

$$A_1(x) = \frac{(m+1)}{k+(n-r-1)(m+1)} \frac{g'_{j|r}(x)}{[g_{j|r}(x) - g_{j|r+1}(x)]}$$

and $$\overline{F}(x) = \exp\left(-\frac{1}{\gamma_{r+1}} \int_{\alpha}^{x} \frac{g'_{j|r}(t)}{[g_{j|r}(t) - g_{j|r+1}(t)]}\,dt\right),$$

from (2.3) as given by Khan *et al.* (2006) and the corresponding result for records is $\overline{F}(x) = \exp\left(-\int_{\alpha}^{x} A_1(t)\, dt \right)$, as obtained by Khan and Athar (2009).

Theorem 2.2: Let $X(i, n, m, k)$, $i = 1, \ldots, n$ be the *gos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support(α, β), and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values $s-1$ and s, $1 \le r < j+1 < s$,

$g_{j|r,l}(x, y) = E[h\{X(j,n,m,k)\} | X(r,n,m,k) = x, X(l,n,m,k) = y]$, $l = s-1, s$

exist, where $g(\ )$ is a finite and differentiable function of y, then

$$\frac{\{\overline{F}(y)\}^{m+1}}{\{\overline{F}(x)\}^{m+1}} = 1 - e^{-I_2} , \qquad m \ne -1, \tag{2.7}$$

where

$$A_2(x, y) = \frac{\frac{\partial}{\partial y} g_{j|r,s}(x, y)}{(s - r - 1)\,[g_{j|r,s-1}(x, y) - g_{j|r,s}(x, y)]} \tag{2.8}$$

and

$$I_2 = \int_{y}^{\beta} A_2(x, t)\, dt \tag{2.9}$$

Proof: We have

$$g_{j|r,s}(x, y) = E[h\{X(j,n,m,k)\} | X(r,n,m,k) = x, X(s,n,m,k) = y]$$

Let $B(x, y) = [\{\overline{F}(x)\}^{m+1} - \{\overline{F}(y)\}^{m+1}]$, then

$$g_{j|r,s}(x,y)[B(x,y)]^{s-r-1} = \frac{(s-r-1)!\,(m+1)}{(j-r-1)!(s-j-1)!}$$

$$\times \int_x^y h(t)[\bar{F}(t)]^m [B(x,t)]^{j-r-1} [B(t,y)]^{s-j-1} f(t)\,dt$$

Differentiating both the sides $w.r.t.$ y, we have

$$\frac{\partial}{\partial y} g_{j|r,s}(x,y)[B(x,y)]^{s-r-1} + g_{j|r,s}(x,y)(s-r-1)(m+1)f(y)[\bar{F}(y)]^m [B(x,y)]^{s-r-2}$$

$$= \frac{(s-r-1)!(m+1)^2}{(j-r-1)!(s-j-2)!} \int_x^y h(t)[\bar{F}(t)]^m f(y)[\bar{F}(y)]^m [B(x,t)]^{j-r-1} [B(t,y)]^{s-j-2} f(t)\,dt$$

$$\frac{\partial}{\partial y} g_{j|r,s}(x,y)[B(x,y)]^{s-r-1} + g_{j|r,s}(x,y)(s-r-1)(m+1)f(y)[\bar{F}(y)]^m [B(x,y)]^{s-r-2}$$

$$= (s-r-1)(m+1) f(y)[\bar{F}(y)]^m [B(x,y)]^{s-r-2} g_{j|r,s-1}(x,y)$$

That is,

$$\frac{(m+1)f(y)[\bar{F}(y)]^m}{B(x,y)} = \frac{\frac{\partial}{\partial y} g_{j|r,s}(x,y)}{(s-r-1)\,[g_{j|r,s-1}(x,y) - g_{j|r,s}(x,y)]} = A_2(x,y)$$

Integrating both the sides $w.r.t.$ y over (y,β), we have

$$\log B(x,y)\big|_y^\beta = -\int_y^\beta A_2(x,t)\,dt\,,$$

implying that

$$\log\left[\frac{\{\bar{F}(x)\}^{m+1} - \{\bar{F}(y)\}^{m+1}}{\{\bar{F}(x)\}^{m+1}}\right] = -\int_y^\beta A_2(x,t)\,dt\,,$$

and hence the result.

Remark 2.2: With the convention $X(0,n,m,k)=x=\alpha$, we have at $r=0$, from (2.8)

$$A_2(y)=\frac{g'_{j|s}(y)}{(s-1)\,[g_{j|s-1}(y)-g_{j|s}(y)]},$$

$$[\bar{F}(x)]^{m+1}=1-\exp\left(-\int_x^{\beta}A_2(t)\,dt\right)\quad,\quad m\neq-1,$$

where $g_{j|s}(y)=E[h(X(j,n,m,k))\,|\,X(s,n,m,k)=y]$ and $\bar{F}(\alpha)=1$, as given by Khan *et al.* (2006).

In Theorem 2.1 we have assumed that $g(x,y)$ is differentiable *w.r.t.* x, whereas in Theorem 2.2 $g(x,y)$ is assumed to be differentiable *w.r.t.* y. However if $g(x,y)$ is assumed to be differentiable *w.r.t.* both x and y, we may combine Theorem 2.1 and Theorem 2.2 as:

Theorem 2.3: Under the assumptions given in Theorem 2.1 and Theorem 2.2,

$$1-\{\bar{F}(x)\}^{m+1}=\frac{e^{I_1}-1}{e^{I_1}+e^{I_2}-1}\,,\qquad m\neq-1 \tag{2.10}$$

and

$$1-\{\bar{F}(y)\}^{m+1}=\frac{e^{I_1}}{e^{I_1}+e^{I_2}-1}\,,\qquad m\neq-1 \tag{2.11}$$

where I_1 and I_2 are as defined in (2.5) and (2.9).

Proof: In view of (2.3) and (2.7), we get

$$e^{I_1} = \frac{1-\{\overline{F}(y)\}^{m+1}}{\{\overline{F}(x)\}^{m+1} - \{\overline{F}(y)\}^{m+1}}$$

and $$e^{I_1} + e^{I_2} - 1 = \frac{1}{\{\overline{F}(x)\}^{m+1} - \{\overline{F}(y)\}^{m+1}}$$

and hence the result.

Now we deduce Theorems for specific family of distributions.

Corollary 2.1:

$$g_{j|r,s}(x,y) = \frac{(s-j)h(x) + (j-r)h(y)}{(s-r)} \tag{2.12}$$

if and only if

$$1 - \{\overline{F}(x)\}^{m+1} = ah(x) + b \quad , \quad m \neq -1 \ , \tag{2.13}$$

where a and b is chosen in such a way that $F(x)$ is a *df.*

Proof: First we will prove that (2.13) implies (2.12).

$$g_{j|r,s}(x,y) = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!}$$
$$\times \int_x^y h(t) \left[\frac{h(t)-h(x)}{h(y)-h(x)}\right]^{j-r-1} \left[1 - \frac{h(t)-h(x)}{h(y)-h(x)}\right]^{s-j-1} \frac{h'(t)}{h(y)-h(x)} dt$$

Now set $u = \left[\frac{h(t)-h(x)}{h(y)-h(x)}\right]$, to get

$$g_{j|r,s}(x,y) = \frac{(s-r-1)!}{(j-r-1)!(s-j-1)!}$$
$$\times \int_0^1 [u\{h(y)-h(x)\} + h(x)]\, u^{j-r-1} (1-u)^{s-j-1} du$$

implying that

$$g_{j|r,s}(x,y)=\frac{(s-j)h(x)+(j-r)h(y)}{(s-r)}$$

To see that (2.12) implies (2.13), we have

$$A_1(x,y)=\frac{h'(x)}{h(y)-h(x)}\,,\quad I_1=-\log\left(1-\frac{\{ah(x)+b\}}{\{ah(y)+b\}}\right)$$

$$A_2(x,y)=\frac{h'(y)}{h(y)-h(x)}\,,\quad I_2=-\log\left(1-\frac{1-\{ah(y)+b\}}{1-\{ah(x)+b\}}\right)$$

Therefore, from Theorem 2.3,

$$1-\{\overline{F}(x)\}^{m+1}=\frac{e^{I_1}-1}{e^{I_1}+e^{I_2}-1}=[ah(x)+b]\,,\quad m\neq-1$$

$$1-\{\overline{F}(y)\}^{m+1}=\frac{e^{I_1}}{e^{I_1}+e^{I_2}-1}=[ah(y)+b]\,,\quad m\neq-1$$

also from Theorem 2.1,

$$\frac{1-\{\overline{F}(x)\}^{m+1}}{1-\{\overline{F}(y)\}^{m+1}}=1-e^{-I_1}=\frac{[ah(x)+b]}{[ah(y)+b]}\,,\qquad m\neq-1,$$

and from Theorem 2.2,

$$\frac{\{\overline{F}(y)\}^{m+1}}{\{\overline{F}(x)\}^{m+1}}=1-e^{-I_2}=\frac{1-[ah(y)+b]}{1-[ah(x)+b]}\,,\qquad m\neq-1,$$

implying that

$$1-\{\overline{F}(x)\}^{m+1}=ah(x)+b\,,\quad m\neq-1,$$

and $\quad -\log\overline{F}(x)=ah(x)+b\,,\qquad m=-1.$

3. Examples for adjacent generalized order statistics

We have

$$g_{r|r,s}(x,y) = E[h\{X(r,n,m,k)\}|X(r,n,m,k) = x, X(s,n,m,k) = y] = h(x)$$

and

$$g_{s|r,s}(x,y) = E[h\{X(s,n,m,k)\}|X(r,n,m,k) = x, X(s,n,m,k) = y] = h(y)$$

Therefore

$$g_{r+1|r,r+2}(x,y) = E[h\{X(j,n,m,k)\}|x \le X \le y] = m(x,y)$$

Also,

$$g_{r+1|r+1,r+2}(x,y) = h(x)$$

$$g_{r+1\,|\,r,\,r+1}(x,y) = h(y)$$

Thus at $j = r+1$ and $s = r+2$

$$A_1(x,y) = \frac{\frac{\partial}{\partial x} m(x,y)}{[m(x,y) - h(x)]}$$

and

$$A_2(x,y) = \frac{\frac{\partial}{\partial y} m(x,y)}{[h(y) - m(x,y)]}$$

Corollary 3.1:

$$m(x,y) = \frac{c}{a(c+1)} \frac{[ah(y)+b]^{c+1} - [ah(x)+b]^{c+1}}{[ah(y)+b]^{c} - [ah(x)+b]^{c}} - \frac{b}{a}, \; c \ne -1 \tag{3.1}$$

if and only if

$$1 - \{\bar{F}(x)\}^{m+1} = [ah(x)+b]^c \, , \quad m \ne -1 \tag{3.2}$$

where a, b, c and $h(x)$ are so chosen that $F(x)$ is a *df.*

Proof: To prove (3.2) implies (3.1), we have

$$m(x,y)=(m+1)\int_x^y \frac{h(t)[\bar{F}(t)]^m f(t)}{[1-\{\bar{F}(y)\}^{m+1}]-[1-\{\bar{F}(x)\}^{m+1}]}\,dt$$

$$=\frac{c}{B(x,y)}\int_x^y [ah(t)+b]^{c-1}h(t)h'(t)\,dt,$$

$$=\frac{c}{B(x,y)}\int_{ah(x)+b}^{ah(y)+b} u^{c-1}\left(\frac{u-b}{a}\right)du$$

$$=\frac{c}{a(c+1)}\frac{[ah(y)+b]^{c+1}-[ah(x)+b]^{c+1}}{[ah(y)+b]^{c}-[ah(x)+b]^{c}}-\frac{b}{a}$$

where, $B(x,y)=[1-\{\bar{F}(y)\}^{m+1}]-[1-\{\bar{F}(x)\}^{m+1}]=[ah(y)+b]^c-[ah(x)+b]^c$

Now to prove (3.1) implies (3.2), we have

$$A_1(x,y)=\frac{\frac{\partial}{\partial x}m(x,y)}{[m(x,y)-h(x)]}=\frac{ac\,h'(x)\,\{ah(x)+b\}^{c-1}}{[\{ah(y)+b\}^c-\{ah(x)+b\}^c]}$$

$$I_1=\int_\alpha^x A_1(t,y)\,dt=-\log\left(1-\frac{\{ah(x)+b\}^c}{\{ah(y)+b\}^c}\right)$$

$$A_2(x,y)=\frac{\frac{\partial}{\partial y}m(x,y)}{[h(y)-m(x,y)]}=\frac{ac\,h'(y)\,[ah(y)+b]^{c-1}}{[\{ah(y)+b\}^c-\{ah(x)+b\}^c]}$$

and

$$I_2=\int_y^\beta A_2(x,t)\,dt=-\log\left(1-\frac{1-\{ah(y)+b\}^c}{1-\{ah(x)+b\}^c}\right)$$

Therefore, in view of the Theorem 2.1,

$$\frac{1-\{\overline{F}(x)\}^{m+1}}{1-\{\overline{F}(y)\}^{m+1}}=1-e^{-I_1}=\frac{\{ah(x)+b\}^c}{\{ah(y)+b\}^c}$$

implying that

$$1-\{\overline{F}(x)\}^{m+1}=K[ah(x)+b]^c,$$

where K is a normalizing constant, but $F(\beta)=1$.

Thus,

$$1-\{\overline{F}(x)\}^{m+1}=[ah(x)+b]^c, \quad m\neq -1.$$

This can also be shown using Theorems 2.3.

Corollary 3.2:

$$m(x,y)=\frac{c}{a(c+1)}\frac{[ah(x)+b]^{c+1}-[ah(y)+b]^{c+1}}{[ah(x)+b]^c-[ah(y)+b]^c}-\frac{b}{a},\ c\neq -1 \tag{3.3}$$

if and only if

$$\{\overline{F}(x)\}^{m+1}=[ah(x)+b]^c, \quad m\neq -1, \tag{3.4}$$

where a,b,c and $h(x)$ are so chosen that $F(x)$ is a *df*.

Proof: First we will prove that (3.4) implies (3.3)

$$m(x,y)=(m+1)\int_x^y\frac{h(t)[\overline{F}(t)]^m f(t)}{[\{\overline{F}(x)\}^{m+1}-\{\overline{F}(y)\}^{m+1}]}dt$$

$$=\frac{c}{B(x,y)}\int_x^y[ah(t)+b]^{c-1}h(t)h'(t)\ dt,$$

$$=\frac{c}{a\ B(x,y)}\int_{ah(x)+b}^{ah(y)+b}u^{c-1}\left(\frac{u-b}{a}\right)du$$

$$= \frac{c}{a(c+1)} \frac{[ah(y)+b]^{c+1} - [ah(x)+b]^{c+1}}{[ah(y)+b]^{c} - [ah(x)+b]^{c}} - \frac{b}{a}$$

where $B(x,y) = [\bar{F}(x)]^{m+1} - [\bar{F}(y)]^{m+1} = [ah(x)+b]^c - [ah(y)+b]^c$.

Now to prove (3.3) implies (3.4), we have

$$A_1(x,y) = -\frac{ac\,h'(x)\,\{ah(x)+b\}^{c-1}}{[\{ah(x)+b\}^c - \{ah(y)+b\}^c]},$$

$$I_1 = -\log\left(1 - \frac{1-\{ah(x)+b\}^c}{1-\{ah(y)+b\}^c}\right)$$

$$A_2(x,y) = -\frac{ac\,h'(y)\,\{ah(y)+b\}^{c-1}}{[\{ah(x)+b\}^c - \{ah(y)+b\}^c]},$$

$$I_2 = -\log\left(1 - \frac{\{ah(y)+b\}^c}{\{ah(x)+b\}^c}\right).$$

Now using Theorem 2.2,

$$\frac{\{\bar{F}(y)\}^{m+1}}{\{\bar{F}(x)\}^{m+1}} = \frac{\{ah(y)+b\}^c}{\{ah(x)+b\}^c}$$

implying that

$$\{\bar{F}(x)\}^{m+1} = K[ah(x)+b]^c,$$

where K is a normalizing constant and $\bar{F}(\alpha) = 1$.

Thus,

$$\{\bar{F}(x)\}^{m+1} = [ah(x)+b]^c, \quad m \neq -1.$$

This may also be proved using Theorem 2.3.

Some examples: $\bar{F}(x)=[ah(x)+b]^{\theta}$, where $\theta=\dfrac{c}{m+1}$, $m\neq -1$.

(a) Power Function Distribution

$$\bar{F}(x)=\left(\frac{\nu-x}{\nu-\mu}\right)^{\theta}=\left(-\frac{1}{\nu-\mu}x+\frac{\nu}{\nu-\mu}\right)^{\theta},\ \mu\le x\le \nu.$$

$$a=-\frac{1}{\nu-\mu},\quad b=\frac{\nu}{\nu-\mu},\quad h(x)=x \text{ and } \theta>0.$$

(b) Pareto Distribution

$$\bar{F}(x)=\left(\frac{\mu+\delta}{x+\delta}\right)^{-\theta}=\left(\frac{1}{\mu+\delta}x-\frac{\delta}{\mu+\delta}\right)^{\theta},\ \mu\le x<\infty.$$

$$a=\frac{1}{\mu+\delta},\quad b=\frac{\delta}{\mu+\delta},\quad h(x)=x \text{ and } \theta<0.$$

(c) Exponential Distribution

$$\bar{F}(x)=\exp[-\lambda(x-\mu)]=\left(1-\frac{\lambda(x-\mu)}{\theta}\right)^{\theta},\quad x\ge\mu$$

$$a=-\frac{\lambda}{\theta},\quad b=\frac{\theta+\lambda\mu}{\theta},\quad h(x)=x \text{ and } \theta\to\infty.$$

More examples with the proper choice of a, b and c may be constructed, as given in Chapter II.

Corollary 3.3:

$$m(x,y)=\frac{c}{(c-1)}\frac{[\{h(x)\}^{c-1}-\{h(y)\}^{c-1}]\,h(x)h(y)}{[h(x)]^{c}-[h(y)]^{c}},\ c\neq 1 \tag{3.5}$$

if and only if

$$1-\{\bar{F}(x)\}^{m+1}=a[h(x)]^{-c}+b,\quad m\neq -1 \tag{3.6}$$

where a, b, c and $h(x)$ are so chosen that $F(x)$ is a *df*.

Proof: For $1-\{\bar{F}(x)\}^{m+1}=a[h(x)]^{-c}+b$, it is easy to show that

$$m(x,y)=(m+1)\int_x^y \frac{h(t)[\bar{F}(t)]^m f(t)}{[\{\bar{F}(x)\}^{m+1}-\{\bar{F}(y)\}^{m+1}]}dt$$

$$=(m+1)\int_x^y \frac{h(t)[\bar{F}(t)]^m f(t)}{[1-\{\bar{F}(y)\}^{m+1}-[1-\{\bar{F}(x)\}^{m+1}]}dt$$

$$=-\frac{ac}{B(x,y)}\int_x^y [h(t)]^{-c-1}h(t)h'(t)\,dt$$

$$=\frac{c}{(c-1)}\frac{[\{h(x)\}^{c-1}-\{h(y)\}^{c-1}]h(x)h(y)}{[h(x)]^c-[h(y)]^c}.$$

where $B(x,y)$ is as defined in (2.6).

Now to prove (3.5) implies (3.6), we have

$$A_1(x,y)=-\frac{c\,h'(x)\,\{h(y)\}^c}{h(x)\,[\{h(x)\}^c-\{h(y)\}^c]}$$

$$I_1=-\log\left(1-\frac{\{a(h(x))^{-c}+b\}}{\{a(h(y))^{-c}+b\}}\right)$$

$$A_2(x,y)=-\frac{c\,h'(y)\,\{h(x)\}^c}{h(y)\,[\{h(x)\}^c-\{h(y)\}^c]}$$

and

$$I_2=-\log\left(1-\frac{1-\{a(h(y))^{-c}+b\}}{1-\{a(h(x))^{-c}+b\}}\right)$$

Therefore in view of the Theorem 2.1,

$$\frac{1-\{\bar{F}(x)\}^{m+1}}{1-\{\bar{F}(y)\}^{m+1}}=1-e^{-I_1}=\frac{[a(h(x))^{-c}+b]}{[a(h(y))^{-c}+b]}$$

implying that

$$1-\{\bar{F}(x)\}^{m+1}=K[a(h(x))^{-c}+b],$$

where K is a normalizing constant. But since $F(\beta)=1$,

$$1-\{\bar{F}(x)\}^{m+1}=[a(h(x))^{-c}+b],\ \ m\neq -1.$$

Remark 3.1: For $1-\{\bar{F}(x)\}^{m+1}=a[h(x)]^{-c}+b$, we have

(a) At $c=-1$, $g(x,y)=\dfrac{h(x)+h(y)}{2}=$ A.M. (3.7)

(b) At $c=2$, $g(x,y)=\dfrac{1}{\frac{1}{2}\left(\frac{1}{h(x)}+\frac{1}{h(y)}\right)}=$ H.M. (3.8)

(c) At $c=\dfrac{1}{2}$, $g(x,y)=\sqrt{h(x)\,h(y)}=$ G.M (3.9)

Where A.M., H.M. and G.M. are arithmetic mean, harmonic mean and geometric mean respectively. This result is also obtained in Ahsanullah and Beg (2008). For ordinary order statistics, similar results were obtained in Balasubramanian and Dey (1997), Balasubramanian and Beg (1992) and Khan and Athar (2004). For records, results are given in Chapter II.

4. Characterization of distributions when $\gamma_i \neq \gamma_j$, $i, j = 1,\dots,n-1$

The conditional *pdf* of $X(j,n,\tilde{m},k)$ given $X(r,n,\tilde{m},k)=x$ and $X(s,n,\tilde{m},k)=y$, $1 \le r < j < s \le n$ is given by

$$f_{j|r,s}(t|x,y) = \frac{\left[\sum\limits_{i=r+1}^{j} a_i^{(r)}(j)\left\{\frac{\bar{F}(t)}{\bar{F}(x)}\right\}^{\gamma_i}\right]\left[\sum\limits_{i=j+1}^{s} a_i^{(j)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(t)}\right\}^{\gamma_i}\right]\frac{f(t)}{\bar{F}(t)}}{\left[\sum\limits_{i=r+1}^{s} a_i^{(r)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(x)}\right\}^{\gamma_i}\right]}$$

Theorem 4.1: Let $X(i,n,\tilde{m},k)$, $i=1,\dots,n$ be the *gos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support (α,β), and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values r and $r+1$, $1 \le r < j-1 < s$,

$g_{j|l,s}(x,y) = E[h\{X(j,n,\tilde{m},k)\}|X(l,n,\tilde{m},k)=x, X(s,n,\tilde{m},k)=y]$, $l=r,r+1$,

then

$$\gamma_{r+1}\frac{f(x)}{\bar{F}(x)} - \frac{\frac{\partial}{\partial x}\mathrm{B}_r^s(x,y)}{\mathrm{B}_r^s(x,y)} = \frac{\frac{\partial}{\partial x}g_{j|r,s}(x,y)}{[g_{j|r,s}(x,y) - g_{j|r+1,s}(x,y)]} \tag{4.1}$$

and

$$\frac{[\bar{F}(x)]^{\gamma_{r+1}}\,\mathrm{B}_r^s(x,y)}{\mathrm{B}_r^s(\alpha,y)} = \exp\left[-\int_{\alpha}^{x} D_1(t,y)\,dt\right], \tag{4.2}$$

where

$$\mathrm{B}_r^s(x,y) = \left[\sum_{i=r+1}^{s} a_i^{(r)}(s)\left\{\frac{\bar{F}(y)}{\bar{F}(x)}\right\}^{\gamma_i}\right], \tag{4.3}$$

and

$$D_1(x,y)=\frac{\frac{\partial}{\partial x}g_{j\mid r,s}(x,y)}{[g_{j\mid r,s}(x,y)-g_{j\mid r+1,s}(x,y)]} \tag{4.4}$$

Proof: We have,

$$g_{j\mid r,s}(x,y)\,\mathrm{B}_r^s(x,y)=\int_x^y h(t)\,\mathrm{B}_r^j(x,t)\,\mathrm{B}_j^s(t,y)\,\frac{f(t)}{\overline{F}(t)}\,dt \tag{4.5}$$

Differentiate both the sides *w.r.t.* x, to get

$$\frac{\partial}{\partial x}g_{j\mid r,s}(x,y)\mathrm{B}_r^s(x,y)+g_{j\mid r,s}(x,y)\left[\frac{\partial}{\partial x}\mathrm{B}_r^s(x,y)\right]$$

$$=\int_x^y h(t)\left[\frac{\partial}{\partial x}\mathrm{B}_r^j(x,t)\right][\mathrm{B}_j^s(t,y)]\,\frac{f(t)}{\overline{F}(t)}dt \tag{4.6}$$

after noting that $\mathrm{B}_r^s(x,x)=0$, as $\sum_{i=r+1}^{s} a_i^{(r)}(s)=0$ (Khan *et al.*, 2006).

Since $a_i^{(r+1)}(s)=(\gamma_{r+1}-\gamma_i)\,a_i^{(r)}(s)$, $\forall\, i$.

Therefore,

$$\mathrm{B}_{r+1}^s(x,y)=\left[\sum_{i=r+2}^{s} a_i^{(r+1)}(s)\left\{\frac{\overline{F}(y)}{\overline{F}(x)}\right\}^{\gamma_i}\right]=\gamma_{r+1}B_r^s(x,y)-\frac{\overline{F}(x)}{f(x)}\left[\frac{\partial}{\partial x}B_r^s(x,y)\right] \tag{4.7}$$

That is,

$$\frac{f(x)}{\overline{F}(x)}\,\frac{\mathrm{B}_{r+1}^s(x,y)}{\mathrm{B}_r^s(x,y)}=\frac{\frac{\partial}{\partial x}g_{j\mid r,s}(x,y)}{[g_{j\mid r,s}(x,y)-g_{j\mid r+1,s}(x,y)]} \tag{4.8}$$

implying that

$$\gamma_{r+1}\frac{f(x)}{\overline{F}(x)}-\frac{\frac{\partial}{\partial x}\mathrm{B}_r^s(x,y)}{\mathrm{B}_r^s(x,y)}=\frac{\frac{\partial}{\partial x}g_{j\mid r,s}(x,y)}{[g_{j\mid r,s}(x,y)-g_{j\mid r+1,s}(x,y)]}$$

Integrating both the sides *w.r.t.* x over (α,x), we get

$$-\log\overline{F}(t)\Big|_{\alpha}^{x}+\frac{1}{\gamma_{r+1}}\log\left[\sum_{i=r+1}^{s}a_i^{(r)}(s)\left\{\frac{\overline{F}(y)}{\overline{F}(x)}\right\}^{\gamma_i}\right]\Bigg|_{\alpha}^{x}$$

$$=\frac{1}{\gamma_{r+1}}\int_{\alpha}^{x}\frac{\frac{\partial}{\partial x}g_{j\mid r,s}(x,y)}{[g_{j\mid r,s}(x,y)-g_{j\mid r+1,s}(x,y)]}\,dt$$

and hence the result.

Remark 4.1: It may be noted that for $\gamma_i\neq\gamma_j$ and $m_1=\cdots=m_{n-1}=m\neq-1$.

$$a_i^{(r)}(s)=\frac{1}{\prod\limits_{\substack{j=r+1\\ i\neq j}}^{s}(\gamma_j-\gamma_i)}=(-1)^{s-i}\,\frac{1}{(m+1)^{s-r-1}}\frac{1}{(i-r-1)!\,(s-i)!},\tag{4.9}$$

Thus,

$$\frac{[\overline{F}(x)]^{\gamma_{r+1}}\,\mathrm{B}_r^s(x,y)}{\mathrm{B}_r^s(\alpha,y)}$$

$$=\frac{[\overline{F}(x)]^{\gamma_{r+1}}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_s}\frac{1}{(m+1)^{s-r-1}(s-r-1)!}\sum\limits_{i=r+1}^{s}(-1)^{s-i}\frac{(s-r-1)!}{(i-r-1)!(s-i)!}\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_i-\gamma_s}}{[\overline{F}(y)]^{\gamma_s}\frac{1}{(m+1)^{s-r-1}(s-r-1)!}\sum\limits_{i=r+1}^{s}(-1)^{s-i}\frac{(s-r-1)!}{(i-r-1)!(s-i)!}\overline{F}(y)^{\gamma_i-\gamma_s}}$$

$$=\frac{[\bar{F}(x)^{m+1}]^{(s-r-1)}\left[1-\frac{\bar{F}(y)^{m+1}}{\bar{F}(x)^{m+1}}\right]^{(s-r-1)}}{[1-\bar{F}(y)^{m+1}]^{(s-r-1)}}$$

$$=\frac{[\{1-\bar{F}(y)^{m+1}\}-\{1-\bar{F}(x)^{m+1}\}]^{(s-r-1)}}{[1-\bar{F}(y)^{m+1}]^{(s-r-1)}}$$

implying that

$$\frac{1-\{\bar{F}(x)\}^{m+1}}{1-\{\bar{F}(y)\}^{m+1}}=1-\exp\left[-\frac{1}{(s-r-1)}\int_{\alpha}^{x} D_1(t,y)\,dt\right]$$

as obtained in (2.3).

Remark 4.2: At $\gamma_s=0$ i.e. $s=k+n+M_s$, by convention $X(s,n,\tilde{m},k)=y=\beta$, and hence $\bar{F}(y)=0$.

Therefore,

$$g_{j|r}(x)=E[h(X(j,n,\tilde{m},k))\,|\,X(r,n,\tilde{m},k)=x]$$

$$D_1(x)=\frac{g'_{j|r}(x)}{[g_{j|r}(x)-g_{j|r+1}(x)]}$$

and

$$\bar{F}(x)=\exp\left(-\frac{1}{\gamma_{r+1}}\int_{\alpha}^{x}\frac{g'_{j|r}(t)}{[g_{j|r}(t)-g_{j|r+1}(t)]}\,dt\right),$$

as $B_r^s(x,\beta)=0$, from (4.2) as given by Khan *et al.* (2006).

Theorem 4.2: Let $X(i,n,\tilde{m},k)$, $i=1,\ldots,n$ be the *gos* from a continuous population with the *df* $F(x)$ and the *pdf* $f(x)$ over the support(α,β), and $h(t)$ be a monotonic and differentiable function of t. If for two consecutive values $s-1$ and $s, 1\le r<j+1<s$,

$g_{j|r,l}(x,y)=E[h(X(j,n,\tilde{m},k))|\, X(r,n,\tilde{m},k)=x,\, X(l,n,\tilde{m},k)=y]$, $l=s,s-1$

exist, where $g(\;)$ is a finite and differentiable function of y, then

$$\gamma_s\frac{f(y)}{\overline{F}(y)}+\frac{\frac{\partial}{\partial y}\mathrm{B}_r^s(x,y)}{\mathrm{B}_r^s(x,y)}=\frac{\frac{\partial}{\partial y}g_{j|r,s}(x,y)}{[g_{j|r,s-1}(x,y)-g_{j|r,s}(x,y)]} \tag{4.10}$$

and

$$\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_i-\gamma_s}=a_s^{(r)}(s)\exp\left[-\int_y^{\beta}D_2(x,t)\,dt\right], \tag{4.11}$$

where $\mathrm{B}_r^s(x,y)$ is as defined in (4.3),

and

$$D_2(x,y)=\frac{\frac{\partial}{\partial y}g_{j|r,s}(x,y)}{[g_{j|r,s-1}(x,y)-g_{j|r,s}(x,y)]} \tag{4.12}$$

Proof: Since

$$g_{j|r,s}(x,y)\,\mathrm{B}_r^s(x,y)=\int_x^y h(t)\,\mathrm{B}_r^j(x,t)\,\mathrm{B}_j^s(t,y)\,\frac{f(t)}{\overline{F}(t)}\,dt$$

Differentiating both the sides *w.r.t.* y, we get

$$\frac{\partial}{\partial y}g_{j|r,s}(x,y)\mathrm{B}_r^s(x,y)+g_{j|r,s}(x,y)\left[\frac{\partial}{\partial y}\mathrm{B}_r^s(x,y)\right]=\int_x^y h(t)[\mathrm{B}_r^j(x,t)]\left[\frac{\partial}{\partial y}\mathrm{B}_j^s(t,y)\right]\frac{f(t)}{\overline{F}(t)}dt$$

Since $a_i^{(r)}(s-1)=(\gamma_s-\gamma_i)\, a_i^{(r)}(s)\ \forall\, i$ as $\sum\limits_{i=r+1}^{s} a_i^{(r)}(s)=0$ (Khan *et al.*, 2006), we have,

$$\frac{f(y)}{\overline{F}(y)}\,\frac{\mathrm{B}_r^{s-1}(x,y)}{\mathrm{B}_r^{s}(x,y)}=\frac{\frac{\partial}{\partial y}g_{j\,|\,r,s}(x,y)}{[g_{j\,|\,r,s-1}(x,y)-g_{j\,|\,r,s}(x,y)]}, \tag{4.13}$$

where $\mathrm{B}_r^{s-1}(x,y)=\left(\gamma_s\,\mathrm{B}_r^{s}(x,y)+\dfrac{\overline{F}(y)}{f(y)}\dfrac{\partial}{\partial y}\mathrm{B}_r^{s}(x,y)\right)$

That is,

$$\gamma_s\frac{f(y)}{\overline{F}(y)}+\frac{\frac{\partial}{\partial y}\mathrm{B}_r^{s}(x,y)}{\mathrm{B}_r^{s}(x,y)}=\frac{\frac{\partial}{\partial y}g_{j\,|\,r,\,s}(x,y)}{[g_{j\,|\,r,\,s-1}(x,y)-g_{j\,|\,r,\,s}(x,y)]}$$

implying that

$$-\gamma_s\log\overline{F}(\beta)+\gamma_s\log\overline{F}(y)+\log\mathrm{B}_r^{s}(x,\beta)-\log\mathrm{B}_r^{s}(x,y)=\int_y^{\beta}D_2(x,t)\,dt$$ Hence

we get,

$$\sum_{i=r+1}^{s}a_i^{(r)}(s)\left(\frac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_i-\gamma_s}=a_s^{(r)}(s)\,\exp\left[-\int_y^{\beta}D_2(x,t)\,dt\right].$$

Remark 4.3: It may be noted that at $\gamma_i\neq\gamma_j$ but $m_1=\cdots=m_{n-1}=m\neq-1$.

$$\frac{\sum\limits_{i=r+1}^{s}a_i^{(r)}(s)\left(\dfrac{\overline{F}(y)}{\overline{F}(x)}\right)^{\gamma_i-\gamma_s}}{a_s^{(r)}(s)}=\left[1-\frac{\overline{F}(y)^{m+1}}{\overline{F}(x)^{m+1}}\right]^{s-r-1}$$

implying that

$$\frac{\{\bar{F}(y)\}^{m+1}}{\{\bar{F}(x)\}^{m+1}} = 1 - \exp\left(-\frac{1}{(s-r-1)} \int_{y}^{\beta} D_2(t, y)\, dt \right)$$

as obtained in (2.7).

Remark 4.4: With the convention $X(0, n, \tilde{m}, k) = x = \alpha$, we have at $r = 0$, from (4.12)

$$D_2(y) = \frac{g'_{j|s}(y)}{[g_{j|s-1}(y) - g_{j|s}(y)]},$$

implying that

$$\sum_{i=1}^{s} a_i(s)[\bar{F}(x)]^{\gamma_i - \gamma_s} = a_s(s) \exp\left[-\int_{x}^{\beta} D_2(t)\, dt \right],$$

where $g_{j|s}(y) = E[h\{X(j,n,\tilde{m},k)\} | X(s,n,\tilde{m},k) = y]$ and $\bar{F}(\alpha) = 1$, as given by Khan *et al.* (2006).

5. Examples

For adjacent *gos* it can be seen that

$$f_{r+1|r,r+2}(t|x,y) = \frac{(m_{r+1}+1)\{\bar{F}(t)\}^{m_{r+1}} f(t)}{[\{\bar{F}(x)\}^{m_{r+1}+1} - \{\bar{F}(y)\}^{m_{r+1}+1}]}, \tag{5.1}$$

which is the *pdf* of case I, where $m_i = m_j = m \neq -1$, except m is replaced by m_{r+1}.

Therefore all the examples considered in Section 3 are also valid for $\gamma_i \neq \gamma_j$. For more examples, please refer to the Table 2.1 of Chapter II.

Chapter IV

CHARACTERIZATION OF CONTINUOUS DISTRIBUTIONS BY CONDITIONAL VARIANCE OF ORDER STATISICS

1. Introduction

Beg and Kirmani (1978) have shown that the conditional variance of $X_{r+1:n}$ given $X_{r:n}=x$ does not depend on x if and only if X has exponential distribution. Khan and Beg (1987) extended the result and proved that the conditional variance of $X^{p}_{r+1:n}$ given $X_{r:n}=x$ does not depend on x if and only if X has Weibull distribution. We, in the present chapter have characterized general form of continuous distribution by considering conditional variance of function of order statistics.

Let $X_{1:n} \le X_{2:n} \le \ldots \le X_{n:n}$ be the order statistics from a continuous population having *pdf* $f(x)$ and *df* $F(x)$ over the support (α, β). Then the conditional *pdf* of $X_{s:n}$ given $X_{r:n}=x$, $1 \le r < s \le n$, is given by (David and Nagaraja, 2003),

$$f_{s|r}(y|x)=\frac{(n-r)!}{(s-r-1)!\,(n-s)!}\frac{[F(y)-F(x)]^{s-r-1}\,[\bar{F}(y)]^{n-s}}{[\bar{F}(x)]^{n-r}}f(y),$$

$$\alpha < x < y < \beta, \qquad (1.1)$$

and the conditional *pdf* of $X_{r:n}$ given $X_{s:n}=y$ by

$$f_{r|s}(x|y)=\frac{(s-1)!}{(r-1)!\,(s-r-1)!}\frac{[F(y)-F(x)]^{s-r-1}[F(x)]^{r-1}}{[F(y)]^{s-1}}f(x),$$

$$\alpha < x < y < \beta, \qquad (1.2)$$

Part of the result of this chapter has appeared in Khan *et al.* (2008).

2. Characterization of distributions

Before coming to the main result, we prove here a Lemma.

Lemma 2.1: Let the *df* $F(x)$ be twice differentiable on (α, β) and let $h(x)$ be a non-decreasing and twice differentiable function of x such that $h(x) \to 0$ as $x \to \alpha$. Then the solution the differential equation

$$\frac{\overline{F}''(x)}{\overline{F}(x)} + (b-1)\left[\frac{\overline{F}'(x)}{\overline{F}(x)}\right]^2 - \frac{h''(x)}{h'(x)}\left[\frac{\overline{F}'(x)}{\overline{F}(x)}\right] - a^2 b[h'(x)]^2 = 0 \qquad (2.1)$$

is

$$\overline{F}(x) = e^{-ah(x)} \qquad (2.2)$$

for all $x \in (\alpha, \beta)$, where $a > 0$ is a constant.

Proof: Let $\dfrac{\overline{F}'(x)}{\overline{F}(x)} = b\,\dfrac{h'(x)}{t}$,

then $\dfrac{\overline{F}''(x)}{\overline{F}(x)} = b\,\dfrac{h''(x)}{t} + b^2\,\dfrac{[h'(x)]^2}{t^2} - b\,\dfrac{h'(x)}{t^2}\,\dfrac{dt}{dx}$

Putting these values in (2.1), we get

$$\frac{dt}{dx} = h'(x)[b^2 - a^2 t^2] \qquad (2.3)$$

Therefore,

$$\frac{1}{2b}\int\left[\frac{1}{(b-at)} + \frac{1}{(b+at)}\right]dt = \int h'(x)\,dx$$

implying that

$$\frac{b+at}{b-at} = A\,e^{2ab\,h(x)},$$

where $a > 0$ and A is the constant of integration.

Hence,

$$\frac{\bar{F}'(x)}{\bar{F}(x)} = \frac{1}{2b}\left[\frac{2A}{Au-1} - \frac{1}{u}\right]\frac{du}{dx},$$

where $u = e^{2ab\,h(x)}$,

implying that

$$\bar{F}(x) = B\,[Ae^{abh(x)} - e^{-abh(x)}]^{1/b} \tag{2.4}$$

where A and B are constants of integration. Since F is bounded, hence $\bar{F}(x) = e^{-ah(x)}$, in view of the initial conditions on $h(x)$.

Theorem 2.1: Let X be the continuous *rv* with *df* $F(x)$ and the *pdf* $f(x)$ over the support (α,β). Let $E[h(X)]^2$ exist, then for some $0<r<n$,

$$V[h\{X_{r+1:n}\}\,|\,X_{r:n} = x] = \frac{1}{a^2(n-r)^2} \tag{2.5}$$

if and only if

$$\bar{F}(x) = e^{-ah(x)} \tag{2.6}$$

where $a>0$, $h(x)$ is a non-decreasing and twice differentiable function of x such that $h(x)\to 0$ as $x\to\alpha$ and $h(x)\bar{F}(x)\to 0$ as $x\to\beta$.

Proof: For $\bar{F}(x) = e^{-ah(x)}$, it is easy to see that (Khan and Abu-Salih, 1989)

$$E[h\{X_{r+1:n}\}\,|\,X_{r:n} = x] = h(x) + \frac{1}{a(n-r)} \tag{2.7}$$

and

$$E[\{h(X_{r+1:n})\}^2\,\Big|\,X_{r:n} = x] = [h(x)]^2 + \frac{2h(x)}{a(n-r)} + \frac{2}{a^2(n-r)^2}, \tag{2.8}$$

implying that

$$V[h\{X_{r+1:n}\}\,|\,X_{r:n}=x]=\frac{1}{a^2(n-r)^2}$$

Now to prove (2.5) implies (2.6), we have for $b=(n-r)\geq 0$,

$$b[\overline{F}(x)]^b\int_x^{\beta}[h(y)]^2[\overline{F}(y)]^{b-1}f(y)\,dy-b^2\left[\int_x^{\beta}[h(y)][\overline{F}(y)]^{b-1}f(y)\,dy\right]^2$$

$$=\frac{[\overline{F}(x)]^{2b}}{a^2b^2} \tag{2.9}$$

Differentiating (2.9) twice *w.r.t.* x and simplifying, we get

$$a^2b\,h'(x)\int_x^{\beta}h'(y)[\overline{F}(y)]^b\,dy=[\overline{F}(x)]^{b-1}f(x) \tag{2.10}$$

Now differentiating (2.10) again *w.r.t.* x, to get

$$h''(x)\int_x^{\beta}h'(y)[\overline{F}(y)]^b f(y)\,dy-[\overline{F}(x)]^b[h'(x)]^2$$

$$=\frac{1}{a^2b}[-(b-1)\,\{\overline{F}(x)\}^{b-2}\{f(x)\}^2+\{\overline{F}(x)\}^{b-1}f'(x)]$$

That is,

$$\frac{\overline{F}''(x)}{\overline{F}(x)}+(b-1)\left[\frac{\overline{F}'(x)}{\overline{F}(x)}\right]^2-\frac{h''(x)}{h'(x)}\left[\frac{\overline{F}'(x)}{\overline{F}(x)}\right]-a^2b[h'(x)]^2=0$$

Hence,

$$\overline{F}(x)=e^{-ah(x)},$$

in view of the Lemma 2.1.

Table 2.1: Examples based on the distribution function $\bar{F}(x) = e^{-a\,h(x)}$

Distribution	$F(x)$	a	$h(x)$
Exponential	$1-e^{-\theta x}$ $0<x<\infty$	θ	x
Rayleigh	$1-e^{-\theta x^2}$ $0<x<\infty$	θ	x^2
Weibull	$1-e^{-\theta x^p}$ $0<x<\infty$	θ	x^p
Pareto	$1-\left(\frac{x}{\alpha}\right)^{-\theta}$ $\alpha<x<\infty$	θ	$\log\left(\frac{x}{\alpha}\right)$
Lomax	$1-\left[1+\left(\frac{x}{\alpha}\right)\right]^{-p}$ $0<x<\infty$	p	$\log\left[1+\left(\frac{x}{\alpha}\right)\right]$
Beta of the I kind	$1-(1-x)^{\theta}$ $0<x<1$	$-\theta$	$\log(1-x)$
Beta of the II kind	$1-(1+x)^{-1}$ $0<x<\infty$	1	$\log(1+x)$
Log logistic	$1-(1+\theta x^p)^{-1}$ $0<x<\infty$	1	$\log(1+\theta x^p)$
Burr Type IX	$1-\left[1+\frac{c\{(1+e^x)^k-1\}}{2}\right]^{-1}$ $-\infty<x<\infty$	1	$\log\left[1+\frac{c\{(1+e^x)^k-1\}}{2}\right]$

Burr Type XII	$1-(1+\theta x^{p})^{-m}$ $0<x<\infty$	m	$\log(1+\theta x^{p})$

Lemma 2.2: Let the *df* $F(x)$ be twice differentiable on (α,β) and let $h(x)$ be a non-increasing and a twice differentiable function of x such that $h(x)\to 0$ as $x\to\beta$. Then the solution of the differential equation

$$\frac{F''(x)}{F(x)}+(r-1)\left[\frac{F'(x)}{F(x)}\right]^{2}-\frac{h''(x)}{h'(x)}\left[\frac{F'(x)}{F(x)}\right]-a^{2}r[h'(x)]^{2}=0 \tag{2.11}$$

is

$$F(x)=e^{-a\,h(x)} \tag{2.12}$$

for all $x\in(\alpha,\beta)$, where $a>0$ is a constant.

Proof: Let $\dfrac{F'(x)}{F(x)}=r\dfrac{h'(x)}{t}$

then $\dfrac{F''(x)}{F(x)}=r\,\dfrac{h''(x)}{t}+r^{2}\,\dfrac{[h'(x)]^{2}}{t^{2}}-r\,\dfrac{h'(x)}{t^{2}}\,\dfrac{dt}{dx}$

Putting these values in (2.11), we get

$$\frac{dt}{dx}=-h'(x)\,[a^{2}t^{2}-r^{2}] \tag{2.13}$$

Therefore,

$$\frac{1}{2r}\int\left[\frac{1}{(at-r)}-\frac{1}{(at+r)}\right]dt=-\int h'(x)\,dx$$

implying that

$$\frac{at+r}{at-r}=A\,e^{2ar\,h(x)}$$

where $a > 0$, and A is the constant of integration.

Hence

$$\frac{F'(x)}{F(x)} = \frac{1}{2r}\left[\frac{2A}{1+Au} - \frac{1}{u}\right]\frac{du}{dx},$$

where $u = e^{2ar\,h(x)}$.

Thus

$$F(x) = B[e^{-arh(x)} + Ae^{arh(x)}]^{1/r} \tag{2.14}$$

where A and B are constants of integration.

Therefore,

$$F(x) = e^{-a\,h(x)}.$$

Theorem 2.2: Let X be the continuous *rv* with *df* $F(x)$ and the *pdf* $f(x)$ over the support (α, β). Let $E[h(X)]^2$ exists, then for some $0 < r < n$,

$$V[h\{X_{r:n}\} \mid X_{r+1:n} = y] = \frac{1}{a^2 r^2} \tag{2.15}$$

if and only if

$$F(y) = e^{-ah(y)} \tag{2.16}$$

where $a > 0$, $h(y)$ is a non-increasing and twice differentiable function of y such that $h(y) \to 0$ as $y \to \beta$ and $h(y)F(y) \to 0$ as $y \to \alpha$.

Proof: It is easy to see that if $F(y) = e^{-ah(y)}$, then

$$E[h\{X_{r:n}\} \mid X_{r+1:n} = y] = h(y) + \frac{1}{ar} \quad \text{(Khan and Abu-Salih, 1989)} \tag{2.17}$$

and,

$$E[\{h(X_{r:n})\}^2 \mid X_{r+1:n}=y]=[h(y)]^2+\frac{2h(y)}{ar}+\frac{2}{a^2r^2} \qquad (2.18)$$

Now in view of (2.15) and (2.16)

$$V[h\{X_{r:n}\} \mid X_{r+1:n}=y]=\frac{1}{a^2r^2}$$

Now to prove (2.15) implies (2.16), we have

$$r\,[F(y)]^r \int_\alpha^y [h(x)]^2[F(x)]^{r-1}f(x)\,dx - r^2\left[\int_\alpha^y h(x)\,[F(x)]^{r-1}f(x)\,dx\right]^2$$

$$=\frac{[F(y)]^{2r}}{a^2r^2} \qquad (2.19)$$

Differentiating (2.19) both the sides *w.r.t.* y twice and simplifying, we get

$$a^2r\;h'(y)\int_\alpha^y h'(x)[F(x)]^r\;dx=[F(y)]^{r-1}f(y) \qquad (2.20)$$

Now differentiate (2.20) again *w.r.t.* y, to get

$$\frac{h''(y)[F(y)]^{r-1}f(y)}{a^2\,r\,h'(y)}+[F(y)]^r[h'(y)]^2$$

$$=\frac{1}{a^2r}[(r-1)\{F(y)\}^{r-2}\{F'(y)\}^2+\{F(y)\}^{r-1}\{F''(y)\}]$$

Therefore,

$$\frac{F''(y)}{F(y)}+(r-1)\left[\frac{F'(y)}{F(y)}\right]^2-\frac{h''(y)}{h'(y)}\left[\frac{F'(y)}{F(y)}\right]-a^2r[h'(y)]^2=0$$

This is of the form of Lemma 2.2. Hence the Theorem is proved.

Table 2.2: Examples based on the distribution function $F(x)=e^{-a\,h(x)}$

Distribution	$F(x)$	a	$h(x)$
Inverse Weibull	$e^{-\theta x^{-p}}$ $0<x<\infty$	θ	x^{-p}
Power function	$\left(\frac{x}{a}\right)^{p}$ $0<x<a$	$-p$	$\log(x/a)$
Logistic	$(1+e^{-x})^{-1}$ $-\infty<x<\infty$	1	$\log(1+e^{-x})$
Burr Type II	$(1+e^{-x})^{-\theta}$ $-\infty<x<\infty$	θ	$\log(1+e^{-x})$
Burr Type III	$(1+x^{-c})^{-k}$ $0<x<\infty$	k	$\log(1+x^{-c})$
Burr Type IV	$\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]^{-k}$ $0<x<c$	k	$\log\left[1+\left(\frac{c-x}{x}\right)^{1/c}\right]$
Burr Type V	$(1+ce^{-\tan x})^{-k}$ $-\frac{\pi}{2}<x<\frac{\pi}{2}$	k	$\log(1+ce^{-\tan x})$
Burr Type VI	$(1+ce^{-k\sinh x})^{-k}$ $-\infty<x<\infty$	k	$\log(1+ce^{-k\sinh x})$

Burr Type VII	$\left(\frac{1+\tanh x}{2}\right)^k$ $-\infty < x < \infty$	$-k$	$\log\left(\frac{1+\tanh x}{2}\right)$
Burr Type VIII	$\left(\frac{2}{\pi}\tan^{-1} e^x\right)^k$ $-\infty < x < \infty$	$-k$	$\log\left(\frac{2}{\pi}\tan^{-1} e^x\right)$
Burr Type X	$(1-e^{-x^2})^k$ $0 < x < \infty$	$-k$	$\log(1-e^{-x^2})$
Burr Type XI	$(x-\frac{1}{2\pi}\sin 2\pi x)^k$ $0 < x < 1$	$-k$	$\log(x-\frac{1}{2\pi}\sin 2\pi x)$
Extreme value II	$e^{-\left(\frac{\theta}{x}\right)^p}$ $0 < x < \infty$	θ^p	x^{-p}

Chapter V

ON RATIO AND INVERSE MOMENTS OF GENERALIZED ORDER STATISTICS FROM THE BURR DISTRIBUTION

1. Introduction

Khan and Khan (1987) have obtained the recurrence relations for single and product moments of order statistics from the Burr distribution. Khan and Ali (1995) extended the results of Khan and Khan (1987) to include the case of negative and ratio moments of order statistics. Pawlas and Szynal (2001) obtained the recurrence relations for single and product moments of generalized order statistics of the Burr distribution. In this chapter, the ratio and inverse moments of generalized order statistics of the Burr distribution when $m_1 = m_2 = \ldots = m_{n-1} = m \neq -1$ and $\gamma_i \neq \gamma_j, i \neq j = 1, \ldots n-1$, are obtained using hyper-geometric function and some of its deductions are also discussed. For the applications of the Burr distribution, one may refer to Tadikamalla (1980).

A random variable X is said to have the Burr distribution if the *pdf* of X is of the form

$$f(x) = \mu p \theta x^{p-1} [1 + \theta x^p]^{-(\mu+1)} \quad ; \quad x > 0 \ , \ \mu, p, \theta > 0 \tag{1.1}$$

$$= 0 \quad \text{otherwise,}$$

with the corresponding *df*

$$F(x) = 1 - [1 + \theta x^p]^{-\mu} \tag{1.2}$$

Therefore, for the Burr distribution, we have

$$f(x) = \frac{\mu\, p\, \theta\, x^{p-1}}{[1+\theta x^{p}]} \bar{F}(x) \tag{1.3}$$

Let $X(r, n, m, k)$ be the r^{th} gos when $m_i = m_j, i, j = 1, \ldots n-1$ and $X(r, n, \tilde{m}, k)$ be the r^{th} gos when $\gamma_i \neq \gamma_j$, $i \neq j$, with respective *pdf*

$$f_{X(r,n,m,k)}(x) = \frac{c_{r-1}}{(r-1)!(m+1)^{r-1}} [1 - \{\bar{F}(x)\}^{m+1}]^{r-1} [\bar{F}(x)]^{\gamma_r - 1} f(x)$$

and

$$f_{X(r,n,\tilde{m},k)}(x) = c_{r-1} \sum_{i=1}^{r} a_i(r) [\bar{F}(x)]^{\gamma_i - 1} f(x)$$

The joint *pdf* of the gos $X(r, n, m, k)$ and $X(s, n, m, k)$ when $m_i = m_j, i, j = 1, \ldots n-1$, $1 \le r < s \le n$, is given by

$$f_{X(r,n,m,k),X(s,n,m,k)}(x,y) = \frac{c_{s-1}}{(r-1)!(s-r-1)!\,(m+1)^{s-2}} [\bar{F}(x)]^{m}$$
$$\times [1 - \{\bar{F}(x)\}^{m+1}]^{r-1} [\{\bar{F}(x)\}^{m+1} - \{\bar{F}(y)\}^{m+1}]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(x) f(y)$$

and the joint *pdf* of $X(r, n, \tilde{m}, k)$ and $X(s, n, \tilde{m}, k)$, $1 \le r < s \le n$, when $\gamma_i \neq \gamma_j, i \neq j = 1, \ldots n-1$, is given by

$$f_{X(r,n,\tilde{m},k),X(s,n,\tilde{m},k)}(x,y)$$
$$= c_{s-1} \left[\sum_{i=r+1}^{s} a_i^{(r)}(s) \left\{ \frac{\bar{F}(y)}{\bar{F}(x)} \right\}^{\gamma_i} \right] \left[\sum_{i=1}^{r} a_i(r) \{\bar{F}(x)\}^{\gamma_i} \right] \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}$$

Let us denote

$$E[X^{j}(r,n,m,k)] = \alpha^{j}_{r,n,m,k}$$

and $E[X^i(r,n,m,k)X^j(s,n,m,k)] = \alpha^{i,j}_{r,s,n,m,k}$

where $X(r,n,m,k)$ is the r^{th} *gos* when $m_i = m_j = m$,

and

$$E[X^j(r,n,\tilde{m},k)] = \alpha^j_{r,n,\tilde{m},k}$$

and $E[X^i(r,n,\tilde{m},k)\ X^j(s,n,\tilde{m},k)] = \alpha^{i,j}_{r,s,n,\tilde{m},k}$

where $X(r,n,\tilde{m},k)$ is the r^{th} *gos* when $\gamma_i \neq \gamma_j$, $i \neq j$.

2. Calculation of moments when $m_i = m_j$, $i,j = 1,\ldots,n-1$

Single moments

Before coming to the main result, the following lemma is proved.

Lemma 2.1: For the Burr distribution as given in (1.1),

$$\Phi_j(a) = \frac{\theta^{-\left(1+\frac{j}{p}\right)}}{p}\,\mathrm{B}\left(\mu a - \frac{j}{p}, 1 + \frac{j}{p}\right) \tag{2.1}$$

and

$$\Phi_0(a) = \frac{1}{\mu a \theta p}, \tag{2.2}$$

where $\Phi_j(a) = \int_0^\infty \frac{x^{j+p-1}}{(1+\theta x^p)^{\mu a+1}}\,dx$ (2.3)

Proof: We have

$$\Phi_j(a) = \int_0^\infty \frac{x^{j+p-1}}{(1+\theta x^p)^{\mu a+1}}\,dx$$

Set $u = \dfrac{1}{(1+\theta x^{p})}$ to get the result as given in (2.1). To prove (2.2), put $j=0$ in (2.1).

Theorem 2.1: Single moment of the Burr distribution is

$$\alpha_{r,n,m,k}^{j-p} = \frac{(\mu p \theta)\, c_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^{i} \binom{r-1}{i} \Phi_{j-p}(\gamma_{r-i}) \tag{2.4}$$

Proof: We have,

$$\alpha_{r,n,m,k}^{j-p} = \frac{c_{r-1}}{(r-1)!} \int_{0}^{\infty} x^{j-p}\, [\bar{F}(x)]^{k+(n-r)(m+1)-1}\, g_{m}^{r-1}[F(x)]\, f(x)\, dx$$

Now on an application of (1.3), we get

$$\alpha_{r,n,m,k}^{j-p} = \frac{(\mu p \theta)\, c_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^{i} \binom{r-1}{i} \int_{0}^{\infty} \frac{x^{j-1}}{(1+\theta x^{p})^{\mu \gamma_{r-i}+1}}\, dx$$

$$= \frac{(\mu p \theta)\, c_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^{i} \binom{r-1}{i} \Phi_{j-p}(\gamma_{r-i})$$

Remark 2.1: If we put $m=0,\, k=1$ in (2.4), we get the result for order statistics

$$\alpha_{r,n,0,1}^{j-p} = \alpha_{r:n}^{j-p} = (\mu p \theta)\, C_{r:n} \sum_{i=0}^{r-1} (-1)^{i} \binom{r-1}{i} \Phi_{j-p}(n-r+i+1)$$

where

$$C_{r:n} = \frac{n!}{(r-1)!(n-r)!}$$

by noting that $\gamma_{r-i} = n - r + i + 1$ and $c_{r-1} = \dfrac{n!}{(n-r)!}$ as obtained by Khan and Ali (1995).

Remark 2.2: At $j = p$ in (2.4), we have

$$\sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \frac{1}{\gamma_{r-i}} = \frac{(r-1)!\,(m+1)^{r-1}}{c_{r-1}}, \; m \neq -1 \tag{2.5}$$

an identity obtained in Athar *et al.*(2007).

Product moments: First we will prove the following lemmas.

Lemma 2.2: For the Burr distribution as given in (1.1),

$$\Phi_{j,\,l}(a,b) = \frac{\theta^{-(2+\frac{j+l}{p})}}{p\,(j+p)} \mathrm{B}\left(\frac{j+l}{p} + 2, \mu b - \frac{l}{p}\right)$$

$$\times {}_3F_2\left[\frac{j}{p}+1, 1-\mu a+\frac{j}{p}, \frac{j+l}{p}+2 \,;\, \frac{j}{p}+2, \mu b+\frac{j}{p}+2 \,;\, 1\right] \tag{2.6}$$

where

$$\Phi_{j,l}(a,b) = \int_0^\infty \int_0^y \frac{x^{j+p-1}}{(1+\theta x^p)^{\mu a+1}} \frac{y^{l+p-1}}{(1+\theta y^p)^{\mu b+1}}\, dx\, dy \tag{2.7}$$

and

$${}_pF_q[a_1,\cdots,a_p\,;b_1,\cdots,b_q\,;1] = \sum_{r=0}^{\infty} \left[\prod_{j=1}^{p} \frac{\Gamma(a_j + r)}{\Gamma(a_j)}\right] \left[\prod_{j=1}^{q} \frac{\Gamma(b_j)}{\Gamma(b_j + r)}\right] \frac{1}{r!}$$

for $p = q+1$ and $\sum_{j=1}^{q} b_j - \sum_{j=1}^{p} a_j > 0$ (Mathai and Saxena, 1973).

Proof: We have,

$$\Phi_{j,l}(a,b)=\int_0^\infty \frac{y^{l+p-1}}{(1+\theta y^p)^{\mu b+1}}\left[\int_0^y \frac{x^{j+p-1}}{(1+\theta x^p)^{\mu a+1}}dx\right]dy$$

Let

$$B(y)=\int_0^y \frac{x^{j+p-1}}{(1+\theta x^p)^{\mu a+1}}dx$$

$$=\frac{1}{p\theta^{\left(1+\frac{j}{p}\right)}}\int_0^{\frac{\theta y^p}{(1+\theta y^p)}} u^{\frac{j}{p}}(1-u)^{\mu a-\frac{j}{p}-1}du$$

$$=\frac{\theta^{-\left(1+\frac{j}{p}\right)}}{p}\mathrm{B}_{\frac{\theta y^p}{(1+\theta y^p)}}\left(\frac{j}{p}+1,\mu a-\frac{j}{p}\right)$$

Thus

$$\Phi_{j,l}(a,b)=\frac{\theta^{-\left(1+\frac{j}{p}\right)}}{p}\int_0^\infty \frac{y^{l+p-1}}{(1+\theta y^p)^{\mu b+1}}\mathrm{B}_{\frac{\theta y^p}{1+\theta y^p}}\left(\frac{j}{p}+1,\mu a-\frac{j}{p}\right)dy, \tag{2.8}$$

where

$$\mathrm{B}_x(p,q)=\int_0^x u^{p-1}(1-u)^{q-1}du. \tag{2.9}$$

Now we know that (Mathai and Saxena, 1973),

$$\mathrm{B}_x(p,q)=p^{-1}x^p\,{}_2F_1(p,1-q;\,p+1;\,x), \tag{2.10}$$

and

$$\int_0^1 u^{a-1}(1-u)^{b-1}\,{}_2F_1(c,d;e;x)\,du = \mathrm{B}(a,b)\,{}_3F_2(c,d,a;e,a+b;1)$$

(2.11)

Substituting these results in (2.8), we get

$$\Phi_{j,l}(a,b)=\frac{\theta^{-\left(1+\frac{j}{p}\right)}}{p}\int_0^\infty \frac{y^{l+p-1}}{(1+\theta y^p)^{\mu b+1}}\left(\frac{\theta y^p}{1+\theta y^p}\right)^{\frac{j}{p}+1}\left(\frac{j}{p}+1\right)^{-1}$$

$$\times{}_2F_1\left[\frac{j}{p}+1,1-\mu a+\frac{j}{p};\frac{j}{p}+2;\frac{\theta y^p}{1+\theta y^p}\right]dy$$

Set $t=\dfrac{\theta y^p}{1+\theta y^p}$, to get

$$\Phi_{j,l}(a,b)=\frac{\theta^{-\left(2+\frac{j+l}{p}\right)}}{p(j+p)}\int_0^1 t^{\frac{j+l}{p}+1}(1-t)^{\mu b-\frac{l}{p}-1}\,{}_2F_1\left[\frac{j}{p}+1,1-\mu a+\frac{j}{p};\frac{j}{p}+2;t\right]dt$$

$$=\frac{\theta^{-(2+\frac{j+l}{p})}}{p\,(j+p)}\,\mathrm{B}\left(\frac{j+l}{p}+2,\mu b-\frac{l}{p}\right)$$

$$\times{}_3F_2\left[\frac{j}{p}+1,1-\mu a+\frac{j}{p},\frac{j+l}{p}+2\,;\frac{j}{p}+2,\mu b+\frac{j}{p}+2\,;1\right]$$

Lemma 2.3: For the Burr distribution,

$$\Phi_{0,l}\,(a,b)=\frac{1}{\mu a\theta p}\,[\Phi_l\,(b)-\Phi_l\,(a+b)] \tag{2.12}$$

and

$$\Phi_{j,0}(a,b) = \frac{1}{\mu b \theta p}\,[\Phi_j(a+b)], \tag{2.13}$$

where $\Phi_j(a)$ is as defined in (2. 3),

and

$$\Phi_{0,0}(a,b) = \frac{1}{(\mu p \theta)^2}\,\frac{1}{b(a+b)} \tag{2.14}$$

Proof: Set $j = 0$ in (2.7), to get

$$\Phi_{0,l}(a,b) = \int_0^\infty \frac{y^{l+p-1}}{(1+\theta y^p)^{\mu b+1}} \left[\int_0^y \frac{x^{p-1}}{(1+\theta x^p)^{\mu a+1}} dx \right] dy$$

$$= \frac{1}{\mu a \theta p} \int_0^\infty \frac{y^{l+p-1}}{(1+\theta y^p)^{\mu b+1}} \left[1 - \frac{1}{(1+\theta y^p)^{\mu a}} \right] dy$$

$$= \frac{1}{\mu a \theta p}\,[\Phi_l(b) - \Phi_l(a+b)]$$

Equation (2.13) and (2.14) can be proved by noting that

$${}_3F_2(a,b,c;c,d;1) = {}_2F_1(a,b;d;1) = \frac{\Gamma d\ \Gamma(d-a-b)}{\Gamma(d-a)\,\Gamma(d-b)}.$$

Theorem 2.2: The generalized product moment of the Burr distribution is given as

$$\alpha_{r,s,n,m,k}^{j,\,l-p} = \frac{(\mu p \theta)^2 c_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times \sum_{i=0}^{r-1} \sum_{t=0}^{s-r-1} (-1)^{i+t} \binom{r-1}{i} \binom{s-r-1}{t} \Phi_{j,l-p}[\gamma_{r-i} - \gamma_{s-t}, \gamma_{s-t}]$$

$$m \neq -1,\ 1 \leq r < s \leq n, \tag{2.15}$$

Proof: Since we have,

$$\alpha_{r,s,n,m,k}^{j,\,l-p} = \frac{c_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \int_0^\infty \int_0^y x^j\, y^{l-p} [\bar{F}(x)^m][1-\bar{F}(x)^{m+1}]^{r-1}$$

$$\times [\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}]^{s-r-1} [\bar{F}(y)]^{\gamma_s - 1} f(x) f(y)\, dx dy$$

Now in the view of (1.3),

$$\alpha_{r,s,n,m,k}^{j,\,l-p} = \frac{c_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \int_0^\infty \int_0^y \frac{x^{j+p-1}}{(1+\theta x^p)} \frac{y^{l-1}}{(1+\theta y^p)}$$

$$\times [\bar{F}(x)^{m+1}][1-\bar{F}(x)^{m+1}]^{r-1} [\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}]^{s-r-1} [\bar{F}(y)^{m+1}]^{\frac{\gamma_s}{(m+1)}} dx dy$$

$$= \frac{(\mu p \theta)^2 c_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \int_0^\infty \int_0^y \frac{x^{j+p-1}}{(1+\theta x^p)} \frac{y^{l-1}}{(1+\theta y^p)}$$

$$\times [\bar{F}(x)^{m+1}]^{i+1} [\bar{F}(x)^{m+1} - \bar{F}(y)^{m+1}]^{s-r-1} [\bar{F}(y)^{m+1}]^{\frac{\gamma_s}{(m+1)}} dx dy$$

$$= \frac{(\mu p \theta)^2 c_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times \sum_{i=0}^{r-1} \sum_{t=0}^{s-r-1} (-1)^{i+t} \binom{r-1}{i} \binom{s-r-1}{t} \int_0^\infty \int_0^y \frac{x^{j+p-1}}{(1+\theta x^p)^{\gamma_{r-i} - \gamma_{s-t}}} \frac{y^{l-1}}{(1+\theta y^p)^{\gamma_{s-t}}} dx dy$$

$$= \frac{(\mu p \theta)^2 c_{s-1}}{(r-1)!(s-r-1)!(m+1)^{s-2}}$$

$$\times \sum_{i=0}^{r-1} \sum_{t=0}^{s-r-1} (-1)^{i+t} \binom{r-1}{i} \binom{s-r-1}{t} \Phi_{j,l-p}[\gamma_{r-i} - \gamma_{s-t}, \gamma_{s-t}]$$

$$m \neq -1,\ 1 \le r < s \le n$$

Remark 2.3: At $j=0$, $l=p$ and using (2.5), (2.15) reduces to an identity

$$\sum_{t=0}^{s-r-1} (-1)^t \binom{s-r-1}{t} \frac{1}{\gamma_{s-t}} = \frac{c_{r-1}\,(s-r-1)!\,(m+1)^{s-r-1}}{c_{s-1}}, \quad m \neq -1$$

(2.16)

Remark 2.4: At $m=0$, $k=1$ in (2.15), we get the result for order statistics

$$\alpha_{r,s:n}^{j,\,l-p} = \frac{(\mu p\theta)^2 n!}{(r-1)!(s-r-1)!(n-s)!} \sum_{i=0}^{r-1} \sum_{t=0}^{s-r-1} (-1)^{i+t} \binom{r-1}{i} \binom{s-r-1}{t}$$

$$\times \frac{\theta^{-(1+\frac{j+l}{p})}}{p\,(j+p)} \mathrm{B}\left(\frac{j+l}{p}+1,\ \mu(n-s+t+1)-\frac{l}{p}+1\right)$$

$$\times {}_3F_2\left[\frac{j}{p}+1, 1-\mu(s-r-t+i)+\frac{j}{p}, \frac{j+l}{p}+1\ ;\ \frac{j}{p}+2, \mu(n-s+t+1)+\frac{j}{p}+2\ ; 1\right],$$

as obtained by Khan and Ali (1995).

3. Calculation of moments when $\gamma_i \neq \gamma_j$, $i,j=1,\ldots,n-1$

Single moments

Theorem 3.1: For the Burr distribution given in (1.1),

$$\alpha_{r,n,\tilde{m},k}^{j-p} = (\mu p\theta)\, c_{r-1} \sum_{i=1}^{r} a_i(r)\, \Phi_{j-p}(\gamma_i) \tag{3.1}$$

Proof: We have,

$$\alpha_{r,n,\tilde{m},k}^{j-p} = c_{r-1} \sum_{i=1}^{r} a_i(r) \int_0^\infty x^{j-p} [\bar{F}(x)]^{\gamma_i} \frac{f(x)}{\bar{F}(x)}\, dx$$

$$= (\mu p\theta)\, c_{r-1} \sum_{i=1}^{r} a_i(r) \int_0^\infty \frac{x^{j-1}}{(1+\theta x^p)^{\mu\gamma_i+1}}\, dx$$

Thus in view of (2.3)

$$\alpha^{j-p}_{r,n,\tilde{m},k} = (\mu\, p\, \theta)\; c_{r-1} \sum_{i=1}^{r} a_i(r)\, \Phi_{j-p}(\gamma_i)\,.$$

Remark 3.1: At $j = p$ in (3.1), we have

$$\sum_{i=1}^{r} \frac{1}{\gamma_i(\gamma_j - \gamma_i)} = \frac{1}{\prod_{i=1}^{r} \gamma_i}$$

This identity was given by Khan and Alzaid (2004).

Remark 3.2: At $m_1 = \ldots = m_{n-1} = m$ in (3.1), we have

$$\alpha^{j-p}_{r,n,m,k} = \frac{(\mu\, p\, \theta)\; c_{r-1}}{(r-1)!\,(m+1)^{r-1}} \sum_{i=0}^{r-1} (-1)^i \binom{r-1}{i} \Phi_{j-p}(\gamma_{r-i})\,,$$

as obtained in (2.4)

Product moments

Theorem 3.2: The generalized product moment of Burr distribution is

$$\alpha^{j,\, l-p}_{r,s,n,\tilde{m},k} = E[X^j(r,n,\tilde{m},k)\, X^{l-p}(s,n,\tilde{m},k)]$$

$$= (\mu\, p\, \theta)^2\, c_{s-1} \left[\sum_{t=r+1}^{s} a_t^{(r)}(s) \left(\sum_{i=1}^{r} a_i(r)\, \Phi_{j,l-p}(\gamma_i - \gamma_t\,, \gamma_t) \right) \right] \quad (3.2)$$

Proof: We have,

$$\alpha^{j,\, l-p}_{r,s,n,\tilde{m},k} = c_{s-1} \int_0^\infty \int_0^y x^j\, y^{l-p} \left(\sum_{t=r+1}^{s} a_t^{(r)}(s) \left[\frac{\bar{F}(y)}{\bar{F}(x)} \right]^{\gamma_t} \right) \left(\sum_{i=1}^{r} a_i(r)\, (\bar{F}(x))^{\gamma_i} \right) \frac{f(x)}{\bar{F}(x)} \frac{f(y)}{\bar{F}(y)}\, dx\, dy$$

$$=c_{s-1}\sum_{t=r+1}^{s}a_t^{(r)}(s)\left[\sum_{i=1}^{r}a_i(r)\int_0^{\infty}y^{l-p}(\bar{F}(y))^{\gamma_t}\left(\int_0^{y}x^j\,(\bar{F}(x))^{\gamma_i-\gamma_t}\frac{f(x)}{\bar{F}(x)}\,dx\right)\frac{f(y)}{\bar{F}(y)}dy\right]$$

$$=(\mu p\theta)^2 c_{s-1}\sum_{t=r+1}^{s}a_t^{(r)}(s)\left[\sum_{i=1}^{r}a_i(r)\int_0^{\infty}\frac{y^{l-1}}{(1+\theta y^p)^{\mu\gamma_t+1}}\left(\int_0^{y}\frac{x^{j+p-1}}{(1+\theta x^p)^{\mu(\gamma_i-\gamma_t)+1}}dx\right)dy\right]$$

and hence the result by an application of (2.7).

Corollary 3.1: For single moment of the Burr distribution,

$$\alpha_{s,n,\tilde{m},k}^{l-p}=(\mu\,p\,\theta)c_{s-1}\sum_{i=1}^{s}a_i(s)\,\Phi_{l-p}(\gamma_i) \tag{3.3}$$

as given in Theorem 3.1.

Proof: Putting $j=0$ in (3.2) and using (2.11), we get

$$\alpha_{s,n,\tilde{m},k}^{l-p}=(\mu\,p\,\theta)^2\,c_{s-1}\left[\sum_{t=r+1}^{s}a_t^{(r)}(s)\left(\sum_{i=1}^{r}a_i(r)\,\Phi_{0,l-p}(\gamma_i-\gamma_t,\gamma_t)\right)\right]$$

$$=(\mu\,p\,\theta)\,c_{s-1}\left[\sum_{t=r+1}^{s}\frac{a_t^{(r)}(s)}{(\gamma_i-\gamma_t)}\left(\sum_{i=1}^{r}a_i(r)\,\{\Phi_{l-p}(\gamma_t)-\Phi_{l-p}(\gamma_i)\}\right)\right]$$

Now using the results [Khan and Alzaid (2004), Bieniek and Szynal (2003)],

$$\sum_{i=1}^{r}\frac{a_i(r)}{(\gamma_i-\gamma_j)}=\prod_{\substack{j=1\\ j\neq i}}^{r}\frac{1}{(\gamma_i-\gamma_j)}\,,\quad \gamma_j\neq\gamma_i\qquad 1\le i\le r\le n,$$

and

$$\sum_{i=r+1}^{s}\frac{a_i^{(r)}(s)}{(\gamma_i-\gamma_j)}=\prod_{\substack{j=r+1\\ j\neq i}}^{s}\frac{1}{(\gamma_i-\gamma_j)}\,,\quad \gamma_j\neq\gamma_i\qquad r+1\le i\le s\le n$$

we have,

$$\alpha^{l-p}_{s,n,\tilde{m},k} = (\mu p\theta)\, c_{s-1}\left(\sum_{t=r+1}^{s} a_t^{(r)}(s)\,\Phi_{l-p}(\gamma_t)\right)\left(\sum_{i=1}^{r}\frac{a_i(r)}{(\gamma_i-\gamma_t)}\right)$$

$$-(\mu p\theta)\, c_{s-1}\left(\sum_{i=1}^{r} a_i(r)\,\Phi_{l-p}(\gamma_i)\right)\left(\sum_{t=r+1}^{s}\frac{a_t^{(r)}(s)}{(\gamma_i-\gamma_t)}\right)$$

$$=(\mu p\theta)\, c_{s-1}\left[\left(\sum_{t=r+1}^{s} a_t^{(r)}(s)\Phi_{l-p}(\gamma_t)\right)\left(\prod_{i=1}^{r}\frac{1}{(\gamma_i-\gamma_t)}\right)+\left(\sum_{i=1}^{r} a_i(r)\Phi_{l-p}(\gamma_i)\right)\left(\prod_{t=r+1}^{s}\frac{1}{(\gamma_t-\gamma_i)}\right)\right]$$

and hence the result.

Remark 3.3: If we put $j=0,\, l=p$ in (3.2), we get an identity

$$\sum_{i=1}^{r}\sum_{t=r+1}^{s}\frac{a_i(r)\,a_t^{(r)}(s)}{(\gamma_i)\,(\gamma_t)}=\frac{1}{c_{s-1}}, \tag{3.4}$$

and at $l=p$, (3.4) yields

$$\sum_{i=1}^{r}\frac{a_{(i)}(r)}{\gamma_i}=\frac{1}{c_{r-1}} \tag{3.5}$$

Combining (3.4) and (3.5), we get another identity

$$\sum_{t=r+1}^{s}\frac{a_t^{(r)}(s)}{(\gamma_t)}=\frac{c_{r-1}}{c_{s-1}} \tag{3.6}$$

Remark 3.4: At $m_1=\ldots=m_{n-1}=m$, Theorem 3.2 reduces to Theorem 2.2.

Remark 3.5: Put $\gamma_r = k+n-r+\sum_{i=r}^{l} m_i$, $1\le r\le l\le n, m_i\in N$, in (3.2), then the product moment of progressive type II censored order statistics of Burr distribution can be obtained.

Remark 3.6: The result is more general in the sense that by simply adjusting $l-p$, we can get interesting results. For example if $l-p=-j$, then $\alpha_{r,s,n,\tilde{m},k}^{j,-j}=E\left[\frac{X(r,n,\tilde{m},k)}{X(s,n,\tilde{m},k)}\right]^{j}$ gives the moments of ratio of two *gos*. For $l-p>0$, $\alpha_{r,s,n,\tilde{m},k}^{j,l-p}$ represents product moments, whereas for $l<p$, it is moment of the quotient of two generalized order statistics of different powers.

4. Calculation of mean and variance

Here we have calculated mean and variance for order statistics but the result can be obtained for progressive type II censoring by suitably adjusting m_i.

Mean of order statistics from the Burr distribution

$\mu=2,\ p=2,\ \theta=1$

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$r=1$	0.4909	0.3866	0.3290	0.2913	0.2642	0.2434
$r=2$	1.0799	0.6995	0.5591	0.4798	0.4271	0.3887
$r=3$		1.2701	0.8399	0.6781	0.5854	0.5231
$r=4$			1.4136	0.9477	0.7707	0.6685
$r=5$				1.5300	1.0362	0.8474
$r=6$					1.6288	1.1117
$r=7$						1.7150

Variance of order statistics from the Burr distribution

$\mu = 2,\ p = 2,\ \theta = 1$

	$n=2$	$n=3$	$n=4$	$n=5$	$n=6$	$n=7$
$r=1$	0.0923	0.0505	0.0346	0.0262	0.0211	0.0176
$r=2$	0.5005	0.1107	0.0588	0.0395	0.0296	0.0237
$r=3$		0.5868	0.1231	0.0639	0.0426	0.0317
$r=4$			0.6588	0.1335	0.0683	0.0449
$r=5$				0.7226	0.1428	0.0721
$r=6$					0.7799	0.1510
$r=7$						0.8327

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